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MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

The Geometry of Matrices

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THE GEOMETRY OF MATRICES

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In §§ 1–3 the matrix notation and theory of the stratified locus V_k^k are developed, and two reflexive processes are defined. §§ 4–6 deal with rank and duality. In §§ 7–9 a matrix pencil is interpreted by means of Grace's collineation defined by four $[k-1]$'s in $[2k-1]$. In § 10 constructions are given. §§ 11–13 interpret a non-singular matrix pencil in terms of reflexive operations; § 14 in terms of certain polar operations and nests of spaces. In § 15 these lead to rational normal loci and their osculating systems. §§ 16–19 interpret the minimal indices of a singular pencil in terms of reflexive processes. The minimal vectors are discussed in § 20, and latent loci in § 21, while § 22 reports shortly on the associated invariant theory.

Introduction

The following investigation gives a geometrical interpretation of classical matrix theory by a systematic recourse to higher dimensions. It is shown that all the chief features of a single matrix of order $k \times k$, or of a matrix pencil—the rank, the latent roots, the exponents of the elementary divisors and the two kinds of minimal indices in the case of a singular pencil—can be explained very naturally by a figure in $[2k-1]$ space. This figure consists of four linear spaces A, B, C and D. Each of A, B, C is a $[k-1]$ having no point in common with another, while D is an unrestricted space of the same or of lower dimensions.

It is well known that three such spaces A, B, C which have no point in common are met by ∞^{k-1} straight lines, each line passing through one point a, b, c in each space. These lines in their aggregate form a *scroll* (to use Room's word for such a locus). It is a locus $\mathscr{R}$ of k dimensions and of order k, a V_k^k let us say. The manner in which an arbitrary space $[k'-1]$, where $k' \leqslant k$, meets, or fails to meet, this $\mathscr{R}$ is quite complicated, but it gives a precise analogy to the behaviour of a matrix or matrix pencil. In particular, latent roots of the characteristic equation of a matrix are connected with the cross-ratios of four collinear points on A, B, C, D: and the fact that spaces D exist which fail to meet the scroll $\mathscr{R}$ corresponds to the existence of certain singular matrix pencils.

The elementary divisors were first established by Sylvester (1850–54), and their properties were first fully demonstrated by Weierstrass (1868). The theory of the singular pencil, with its minimal indices of two kinds is due to Kronecker (1874). Segre gave various geometrical accounts of the theory of Weierstrass (Segre 1884*d*, 1887 and further references) and of Kronecker (Segre 1884 *b*, *c*), but in the latter case confined his treatment to that of pencils of general cones whose matrices are necessarily symmetrical and therefore yield identical pairs of minimal indices. I am not aware that the more general and unsymmetrical case has ever before been discussed geometrically in its entirety.

The geometrical treatment that follows is closely akin to that of Predella (1889–92), who brought out the importance of a set of spaces conjugate to those which characterize the elementary divisors. An account of this and indeed of the whole geometrical theory of the homography or collineation is given in the *Geometria Proiettiva degli Iperspazi* by Bertini (1923). I am grateful to Mr W. L. Edge for bringing to my notice several of the geometrical references.

In what follows I work systematically with a matrical analytical geometry, a medium which seems to be the natural link between the algebra and the geometry. I have introduced a word 'medial' for the important self-dual space midway between a point and a prime in space of odd dimensions, and have found it most convenient in practice. I have also used the word 'reflexion' rather than 'projection' for the two processes denoted by $\rightarrow$ and $\twoheadrightarrow$, since they are generally employed alternately, and also because the double-headed arrow process is usually not a one-to-one correspondence.

Medial spaces

1. In odd dimensional space $[2k-1]$ linear spaces $[k-1]$ occupy a central position, that of being self-dual. For this reason it is desirable to give them a special name, and I propose to call them *medial* spaces or briefly medials. Thus a point is a medial on a line, a line is a medial in a plane, and a plane in [5], and so on.

Two medials A and C, which have no point in common, form a *basis*, in the sense that every point of space $[2k-1]$ either belongs to A or C, or else is in line with a point a of A and c of C. In fact the k-fold space containing A and any external point d intersects the $(k-1)$-fold C in one point c: and the straight line cd, lying in the space Ac, must meet A in a point a.

Analytically let $$\{x, y\} = \{x_1, x_2, \ldots, x_k, y_1, y_2, \ldots, y_k\} \tag{1}$$

be the homogeneous co-ordinates of a point of $[2k-1]$, referred to the medials A and C as basis: that is to say, let any point of A be given by $\{x, 0\}$, and any point of C by $\{0, y\}$. Here x is to be regarded as a column vector of k components, as also y, while $\{x, y\}$ denotes a column vector of all $2k$ components in the above order. We shall call $\{x, y\}$ the *matrical co-ordinates* of the point referred to the basis A, C.

This basis is one of ${}_{2k}C_k$ ways of dividing the simplex of reference into two equal sets of k points. In what follows, the matrix notation will be systematically used.

By the *matrical equation* $x = 0$ is understood that all x_i vanish: hence it represents the medial C. Likewise $y = 0$ is the matrical equation of the medial A. If $x \neq 0$, $y \neq 0$ the point $\{x, y\}$, or d, is external to both A and C. Manifestly the three points

$$\{x, 0\}, \quad \{x, y\}, \quad \{0, y\} \tag{2}$$

are collinear, a, d and c let us say: also a and c are the only points of A and C which are in line with this external point d. We shall call a and c the reflexions of this d in the basic medials A and C, and shall denote the relation by a double arrow, thus:

$$a \twoheadrightarrow d \twoheadrightarrow c \quad \text{or} \quad a \twoheadrightarrow c. \tag{3}$$

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Any other point on the line ac is given by $\{\rho x, \sigma y\}$, where ρ and σ are scalar factors. Suppose now that the point $\{x, y\}$ lie on a third medial B. We can define B by k independent points $b_1, b_2, \ldots, b_k$, or by their matrix

$$B = [b_1, b_2, \ldots, b_k] = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} = \{B_1, B_2\}, \tag{4}$$

where B has k columns b_j and $2k$ rows, while B_1 and B_2 denote the square arrays of the first k and last k rows. In this notation A would be $\{I, 0\}$ and C, $\{0, I\}$, where I is the unit matrix of order k.

If θ is any column vector of k elements, it may post-multiply the matrix B, and the result

$$B\theta \quad \text{or} \quad \{B_1\theta, B_2\theta\} \tag{5}$$

is the parametric form of the co-ordinates of any point on this medial. It is useful to write

$$\operatorname{loc}\{B_1\theta, B_2\theta\} \tag{6}$$

to denote B as the locus of the point when the parameter θ varies over its range of values. The same notation is also useful when the parameter enters to higher degree and the locus is curved.

If both B_1 and B_2 are non-singular, no solution of $B_1\theta = 0$ or $B_2\theta = 0$ exists, except $\theta = 0$. Hence no point of B is common to A or C. That is, B is skew to both. On solving $x = B_1\theta$ we then have $\theta = B_1^{-1}x$, which is equivalent to a change of simplex *within* the medial A. If also we write $y = B_2\theta$, $\theta = B_2^{-1}y$ we can express the co-ordinates of the collinear points a, b, c, belonging respectively to A, B, C, by the vectors

$$\{\theta, 0\}, \quad \{\theta, \theta\}, \quad \{0, \theta\}, \tag{7}$$

where $\theta = \{\theta_1, \theta_2, \ldots, \theta_k\}$.

The matrical equation

$$\sigma x = \rho y \tag{8}$$

will be that of the locus $\mathscr{R}$ which consists of all points in the variable straight line abc, given by (7), for all values of the parametrical vector θ. In full this equation yields the system

$$\frac{x_1}{y_1} = \frac{x_2}{y_2} = \ldots = \frac{x_k}{y_k} = \frac{\rho}{\sigma}, \tag{9}$$

which is evidently the locus of the point $\{\rho\theta, \sigma\theta\}$ for all values of the ratio $\rho:\sigma$ and of θ. Thus we can write

$$\mathscr{R}: \ \operatorname{loc}\{\rho\theta, \sigma\theta\}. \tag{10}$$

This locus is a *scroll*, in the sense that it is generated by the ∞^1 linear spaces each of which is given by a value of $\rho:\sigma$. These spaces are determined by k independent equations $\sigma x_i = \rho y_i$ of (9), and are therefore medials. They are often called generators; but they stratify the figure, so to speak, and will here be referred to as the *strata* of $\mathscr{R}$. They include A, B and C in particular. Each point of $\mathscr{R}$ lies in one stratum (as the linearity of (10) implies), and no two strata have a common point. Any three such strata determine the scroll.

Simultaneously the scroll is generated by the ∞^{k-1} straight lines, each of which is given by (10) with a fixed θ and a varying $\rho:\sigma$. These lines, which are often called directrices, will here be called the *generating lines* of $\mathscr{R}$. From (10) it follows that each point of $\mathscr{R}$ lies on one generating line, and that no two such lines have a common point.

The line joining two distinct points $\{x, y\}$ and $\{x', y'\}$ of $\mathscr{R}$ must either lie in a stratum or coincide with a generating line, or meet $\mathscr{R}$ nowhere else. This follows from (10) by finding the condition for the point $\{\nu x+\nu' x', \nu y+\nu' y'\}$ to lie on $\mathscr{R}$.

Any point of $\mathscr{R}$ is specified by the generating line g and the stratum S which meet at the point. We may speak of the totality of strata as the *stratification*, and that of the generating lines as the *regulus* of $\mathscr{R}$. When $k = 2$ the scroll is a quadric surface Q_{12}, $x_1 y_2 = x_2 y_1$, for which the stratification is a second regulus; but this symmetry is lost in higher dimensions. In [5] the strata are planes and the regulus consists of ∞^2 straight lines. Also the locus $\mathscr{R}$ is now $(k>2)$ the complete intersection of all the quadrics Q_{ij}, $x_i y_j = x_j y_i$. Each of these is a quadric primal cone of rank four and signature zero (in the real case), as four effective co-ordinates are involved. Any $k-1$ linearly independent among the $\frac{1}{2}k(k-1)$ cones Q_{ij} will determine the scroll.

The strata relate the points of any two generating lines in a (1, 1) correspondence, and conversely these lines relate the points of any two strata in the same way. Any four strata meet each generating line in four points having the same cross-ratio. In particular, one of the six cross-ratios for the four strata

$$\{x, 0\}, \quad \{x, \lambda x\}, \quad \{0, x\}, \quad \{x, \mu x\} \tag{11}$$

is λ/μ.

By the *line* x is meant the generating line which passes through these points.

Change of basis

2. More generally let $A = \{A_1, A_2\}$, $B = \{B_1, B_2\}$, $C = \{C_1, C_2\}$ be three medials where each of these six matrix components is of order $k\times k$. Each matrix A, B, C will then be of rank k. Let x, y, z denote any three points in them respectively. Then the matrical co-ordinates of these points will be Ax, By, Cz as in § 1 (5). Furthermore, the equation

$$Ax+By+Cz = 0, \tag{1}$$

will hold if and only if these three points are collinear, as is apparent when this is written out in full. Alternatively, we may write two equations

$$A_1 x+B_1 y+C_1 z = 0, \quad A_2 x+B_2 y+C_2 z = 0, \tag{2}$$

involving six $k\times k$ matrices and three column vectors each of k components, instead of one with three $2k\times k$ matrices and the same vectors.

For example: The matrical equation

$$\begin{bmatrix} a_1 & a_1' \\ a_2 & a_2' \\ a_3 & a_3' \\ a_4 & a_4' \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}+\begin{bmatrix} b_1 & b_1' \\ b_2 & b_2' \\ b_3 & b_3' \\ b_4 & b_4' \end{bmatrix}\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}+\begin{bmatrix} c_1 & c_1' \\ c_2 & c_2' \\ c_3 & c_3' \\ c_4 & c_4' \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = 0, \tag{3}$$

specifies that three points x, y, z, which lie respectively on three lines through pairs of points a, a' and b, b' and c, c' of [3] space, are collinear. The four elements a_i are the co-ordinates of the point a referred to any tetrahedron of reference for [3], while x_1 and x_2 are the binary co-ordinates of a point on the line aa' referred to a and a' as base points of reference.

From two such equations (2) we may eliminate either x or y or z very readily. For assuming C_1 and C_2 to be non-singular we have

$$C_1^{-1}A_1x+C_1^{-1}B_1y = -z = C_2^{-1}A_2x+C_2^{-1}B_2y:$$

whence
$$Ly = Mx, \tag{4}$$

where
$$L = C_1^{-1}B_1-C_2^{-1}B_2, \quad M = C_2^{-1}A_2-C_1^{-1}A_1. \tag{5}$$

Equation (4), which involves certain $k\times k$ matrices L and M explicitly, gives the *collineation* between points of A and points of B determined by the transversal lines of the medials A, B, C. When A, B, C are mutually skew these lines generate the scroll $\mathscr{R}$. If L is non-singular we have $y = Qx$, where $Q = L^{-1}M$; and similarly $z = Rx$, say, by eliminating y from (1). The points on A, B, C of a generating line of the scroll are given by Ax, BQx, CRx. In fact the scroll is now

$$\operatorname{loc}\{\rho Ax,\ \sigma CRx\}, \tag{6}$$

with $\rho:\sigma$ and x as parameters of stratum and generating line respectively. This reverts to the previous form § 1 (10) when $A = \{I, 0\}$, $CR = \{0, I\}$.

The parametric form of the scroll, referred to the basis $A = \{I, 0\}$, $C = \{0, I\}$ and containing $\{B_1, B_2\}$ as another stratum, is therefore

$$\operatorname{loc}\{\rho B_1x,\ \sigma B_2x\}, \tag{7}$$

and the points $\{B_1x, 0\}$ of A, $\{0, B_2x\}$ of C are in line with $\{B_1x, B_2x\}$, that is with Bx of B. Such points a and c will be called reflexions of b through B on to A and C, and will be denoted by an arrow relation

$$a\to b\to c \quad \text{or briefly} \quad a\to c. \tag{8}$$

It is important to notice that both of the k-rowed determinants $|B_1|$ and $|B_2|$ are non-zero—otherwise B meets either A or C. This being so the relation of a to c is symmetrical and one-to-one, and we can also write $c\to b\to a$, $c\to a$.

Now let $D = \{D_1, D_2\}$ be a $2k\times k$ matrix of rank r where $0<r\leqslant k$. It represents an $[r-1]$ which is either a medial ($r = k$) or else a lower space. Let it be called a *punctual* space, in contrast to linear spaces $[r'-1]$ where $2k-1>r'\geqslant k$ which are *primary** spaces; so that a medial is in both categories.

Any point of D is given by $\{D_1x, D_2x\}$, or Dx, and is in line with $\{D_1x, 0\}$ of A and $\{0, D_2x\}$ of C. Since D is of rank equal or less than k it is possible for D to meet A or C or both. Whenever a, d, c are points of A, D and C which are both collinear and distinct, we shall call a and c reflexions of d through D on to A and C, and shall express this by a double arrow,

$$a\twoheadrightarrow d\twoheadrightarrow c, \quad \text{or briefly} \quad a\twoheadrightarrow c. \tag{9}$$

* The word *primal* has already become attached to n-dimensional loci of any order in $[n+1]$.

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This relation is no longer necessarily symmetrical, and, as we shall see, will lead by its lack of symmetry to the minimal indices in the case of a singular matrix pencil. We note that, for a given point a of A, there may be an infinity of points d of D, distinct from a, for which ad meets C. When *all* such points of D in line with a and a point of C are taken, together with all the points which are common to D and C, let the resulting locus in C be called C_a. We shall express this by

$$a \twoheadrightarrow D \twoheadrightarrow C_a, \quad \text{or briefly} \quad a \twoheadrightarrow C_a. \tag{10}$$

If both c and c' belong to C_a so must the line cc'. Consequently C_a is a linear subspace of C. This C_a is called the reflexion of a through D to C.

Similarly $c \twoheadrightarrow D \twoheadrightarrow A_c$ gives the reflexion A_c of c to A.

When all possible positions of a within A are taken which are either in D or else in line with some point of D and some point of C, the three points being distinct, the result is again a subspace A_D of A. This is called the *total reflexion* of D on to A. Similarly for C_D, the total reflexion of D on to C.

Finally, when a describes a locus or region ϕ of A, the aggregate of distinct points of all the C_a will form a region C_ϕ of C, and we shall call this the reflexion of ϕ through D to C, namely,

$$\phi \twoheadrightarrow D \twoheadrightarrow C_\phi \quad \text{or} \quad \phi \twoheadrightarrow C_\phi. \tag{11}$$

When no such region C_ϕ exists the result is written

$$\phi \twoheadrightarrow D \twoheadrightarrow 0. \tag{12}$$

For the general position of the $[r-1]$ space D, the dimensions of these reflexions C_a, C_ϕ, C_D are quite complicated. But when D is a medial, such as B, which is skew to both A and C the reflexions are related in $(1, 1)$ correspondence. For example, A_D is then identical with A.

The matrix pencil

3. The usual theory of a single $k \times k$ matrix, or a pair of such, or again a pencil of such, is comprised in the study of the pencil $\rho D_1 + \sigma D_2$. This pencil will be represented geometrically by the locus of the point

$$\{D_1\theta, D_2\theta\}, \quad \theta = \{\theta_1, \theta_2, \ldots, \theta_k\}, \tag{1}$$

which is the space D referred to the basis $A = \{I, 0\}$ and $C = \{0, I\}$. Now any other two of the strata of $\mathscr{R}$ are given by $\overline{A} = \lambda A + \lambda' C$ and $\overline{C} = \mu A + \mu' C$, where the four coefficients are scalar and such that $\lambda\mu' \neq \lambda'\mu$ (otherwise the strata coincide). Also if $\{\xi, \eta\}$ are the matrical co-ordinates of a point $\{x, y\}$ referred to $\overline{A}$, $\overline{C}$ as basis of reference, we shall have the relation $Ax + Cy = \overline{A}\xi + \overline{C}\eta$, that is

$$x = \lambda\xi + \mu\eta, \quad y = \lambda'\xi + \mu'\eta, \tag{2}$$

for ξ, η in terms of the original co-ordinates. Hence D will now be partitioned into $\{\lambda\overline{D}_1 + \mu\overline{D}_2, \lambda'\overline{D}_1 + \mu'\overline{D}_2\} = \{D_1, D_2\}$, and the pencil will become $\overline{\rho}\overline{D}_1 + \overline{\sigma}\overline{D}_2$, where

$$\{\xi, \eta\} = \{\overline{D}_1\theta, \overline{D}_2\theta\} \quad \text{and} \quad \rho = \lambda\overline{\rho} + \lambda'\overline{\sigma}, \quad \sigma = \mu\overline{\rho} + \mu'\overline{\sigma}. \tag{3}$$

But this, which also occurs in the algebraic theory of a matrix pencil, is there called a *change of basis* of the pencil. Hence we have proved the following result:

THEOREM 1. *A $k \times k$ matrix pencil, whether singular or non-singular, can be represented by a linear space of dimension less than k, with reference to a scroll $\mathcal{R}$ and with two of its strata for basis. Algebraic change of basis for the pencil corresponds to change of reference to two other strata.*

RANK OF A MATRIX

4. When a matrix A consisting of m rows and n columns has a rank r, there are exactly $n-r$ linearly independent non-zero solutions for the ratios of $x_1, x_2, \ldots, x_n$ in the system of linear equations given by

$$Ax = 0. \tag{1}$$

This well-known theorem has a useful corollary, as follows. Replace A by the matrix

$$[A, B],$$

wherein A has n_1 and B has n_2 columns, and both have m rows. Also let the ranks of the parts be n_1 and n_2 respectively, while that of the whole is r, so that

$$r \geqslant n_1, \quad r \geqslant n_2, \quad n_1 + n_2 \geqslant r.$$

Let $\{x, y\}$ be a column vector of n_1 components x_i and n_2 components y_i. The expression $[A, B]\{x, y\} = 0$, or

$$Ax + By = 0$$

will evidently represent a system of m linear homogeneous equations in the x_i and y_i: and it will have $n_1 + n_2 - r$ non-zero linearly independent solutions. In each such solution *both x and y are non-zero*: else, if $y = 0$, $x \neq 0$ then $Ax = 0$, where A is of rank n_1; that is, all x would vanish, which is a contradiction. Similarly for any number of such partitions of a matrix by columns.

Using this principle for the case when $n_1 = n_2 = \text{etc.} = p$, we may develop the ideas of § 2. For consider the matrical equations

$$\left.\begin{aligned} Ax &= 0, & q &= 1,\\ Ax + By &= 0, & q &= 2,\\ Ax + By + Cz &= 0, & q &= 3,\\ \ldots\ldots\ldots\ldots &, & \ldots\ldots &, \end{aligned}\right\} \tag{2}$$

where each of $A, B, C, \ldots$ has p columns and rank p, while each of $x, y, z, \ldots$ is a column of p elements. Let each of A, B, C, etc., in the qth equation have pq rows, for each successive value of q. Since the rank of each matrix is p, the first equation has only the solution $x = 0$, while all the rest have non-zero solutions. The first equation is a converse way of stating that there is no point of A external to A; the second states that the point x of A coincides with y of B; the third that x of A, y of B and z of C are collinear; and so on.

In the second equation A and B are medials in odd space $[2p-1]$; in the third A, B, C are trimedials or $[p-1]$-folds in $[3p-1]$. Thus $Ax = \{A_1x, A_2x, A_3x\}$ are the matrical co-ordinates of a point of A, where each A_i is a $p \times p$ matrix. This third equation implies three submatrical equations

$$A_i x + B_i y + C_i z = 0 \quad (i = 1, 2, 3),$$

from which any two of x, y, z can be eliminated. Systematic elimination of z yields, say, $Ly = Mx$, $L_1 y = M_1 x$, so that $(L^{-1}M - L_1^{-1}M_1)\, x = 0$. But $x \neq 0$: hence the p-rowed determinant

$$| L^{-1}M - L_1^{-1}M_1 |$$

must vanish. This determinant is naturally a condensation of the $3p$-rowed determinant $| A_1 B_2 C_3 |$. This means that, when $x \neq 0$, the rank of the matrix $[A, B, C]$ is less than $3p$. By the above corollary on rank, neither y nor z can vanish when $x \neq 0$. Hence three $[p-1]$'s in $[3p-1]$ but not in $[3p-2]$ have no transversal line, but if they lie in $[3p-2]$ but not in $[3p-3]$ they have one such line—answering to the unique solution for $x:y:z$.

This method of elimination is virtually that of A. R. Richardson (1928) and applies to any number of such right- (or left-) handed matrical linear equations. Also the corollary to the theorem on rank affords a ready means of representing the double sets of spaces, which generalize on the double-six of lines, and which have been established by T. G. Room (1929).

In the third equation (2) the rank of $[A, B, C]$ may be $3p$ or less, but not less than p, that of A or B or C. By taking the rank successively equal to p, $p-1$, $p-2$, ... we infer that

three $[p-1]$'s in $[3p-1]$ but not in $[3p-2]$ have no transversal line,
three $[p-1]$'s in $[3p-2]$ but not in $[3p-3]$ have one transversal line,
three $[p-1]$'s in $[3p-3]$ but not in $[3p-4]$ have ∞^1 transversal lines,
three $[p-1]$'s in $[3p-4]$ but not in $[3p-5]$ have ∞^2 transversal lines,
etc.

From $Ax + By + Cz + Dt = 0$, we infer that a plane $xyzt$ traverses A, B, C, D and meets each in one point. Here $q = 4$, and $p \leqslant r \leqslant 4p$. Hence

four $[p-1]$'s in $[4p-1]$ but not in $[4p-2]$ have no transversal plane,
four $[p-1]$'s in $[4p-2]$ but not in $[4p-3]$ have one transversal plane,
etc.

In general from the qth equation we infer that

q $[p-1]$'s in $[pq-1]$ but not in $[pq-2]$ have no transversal $[q-2]$,
q $[p-1]$'s in $[pq-2]$ but not in $[pq-3]$ have one transversal $[q-2]$,
q $[p-1]$'s in $[pq-3]$ but not in $[pq-4]$ have ∞^1 transversal $[q-2]$'s,
etc.

The second result in each of these hierarchies leads to a *double* $q+1$, namely, a *double* $q+1$ of $[p-1]$'s and $[q-2]$'s in $[pq-2]$. For example, $(p = 2,\ q = 3)$ a double set of four lines a, b, c, d and a', b', c', d' in [4] but not in [3], where d' is the single transversal of a, b, c, and so on.

The third result in each hierarchy leads, when $p = 2$, to the scroll $\mathscr{R}$, with $k = q-1$. Here q lines in $[2q-3]$ but not in $[2q-4]$ have ∞^1 transversal $[q-2]$'s, which are the strata. These q lines are $k+1$ of the generating lines, which are just the requisite number to determine the collineation, set up by all the generators, between any two fixed medial strata.

The $(k+1)$th result when $p = k$, $q = 3$ also gives $\mathscr{R}$ in the alternative form:

three $[k-1]$'s in $[2k-1]$ but not in $[2k-2]$ have ∞^{k-1} transversal lines.

Dual properties

5. Answering to the point $\{x, y\}$ is the prime $[u, v]$ where

$$[u, v] \equiv [u_1, u_2, \ldots, u_k, v_1, v_2, \ldots, v_k] \tag{1}$$

are the $2k$ co-ordinates dual to those of a point. The point lies on the prime if

$$\Sigma u_j x_j + \Sigma v_j y_j = 0,$$

that is if

$$ux + vy = 0. \tag{2}$$

Manifestly the row vector $[0, v]$ gives the prime co-ordinates of the medial A, since this prime contains the point $\{x, 0\}$ for all values of v and x. For like reasons the prime co-ordinates of the medials B and C are $[u, -u]$ and $[u, 0]$, corresponding to their point co-ordinates $\{x, x\}$ and $\{0, x\}$ respectively.

Again, if the prime $[u, v]$ contains the stratum loc$\{\rho x, \sigma x\}$ ($\rho : \sigma$ constant), then $[u, v]\{\rho x, \sigma x\} = 0$ for all values of x, so that $\rho u + \sigma v$ must vanish. Hence the expression

$$[\sigma u, -\rho u] \tag{3}$$

gives the prime co-ordinates of the same stratum $\rho : \sigma$ of $\mathscr{R}$: so that the tangential (or prime) equations of $\mathscr{R}$ are

$$\frac{u_1}{v_1} = \frac{u_2}{v_2} = \ldots = \frac{u_k}{v_k} = -\frac{\sigma}{\rho}, \tag{4}$$

which reproduce the form of the point equations (§ 1 (9)). Just as a point which lies on any stratum belongs to $\mathscr{R}$, so we say that a prime which contains any stratum belongs to $\mathscr{R}$. It is readily seen that such a prime contains one and one stratum only of $\mathscr{R}$. And just as $\mathscr{R}$ is the locus of the point $\{\rho x, \sigma x\}$ for varying parameters $\rho : \sigma$ and x, so also it is the envelope of the prime $[\sigma u, -\rho u]$ for varying $\rho : \sigma$ and u; let us say,

$$\text{env}\,[\sigma u, -\rho u]. \tag{5}$$

For a fixed ratio $\rho : \sigma$ we obtain the same stratum from either locus or envelope; but with a fixed u and varying ratio we obtain a *generating secundum* G, that is, a $[2k-3]$ which meets each stratum in a $[k-2]$. This is the dual of the generating line g which meets each stratum in a point.

For example, when $k = 3$, the medials are planes in $[5]$. The strata of $\mathscr{R}$ are planes, the g generators are lines which meet each stratum in a point, while the G generators are $[3]$'s which meet each stratum in a line.

A point $\{D_1\phi, D_2\phi\}$ of a given punctual space D will lie on $\mathscr{R}$, that is on loc $\{\rho x, \sigma x\}$, if $\lambda D_1\phi = \rho x$, $\lambda D_2\phi = \sigma x$ for some value of λ. Hence

$$(\sigma D_1 - \rho D_2)\,\phi = 0 \quad (\phi \neq 0). \tag{6}$$

This can only happen if the determinant $|\,\sigma D_1 - \rho D_2\,|$ vanishes. When D is a medial which meets neither A nor C, this determinant gives a binary $k-ic$ in $\rho:\sigma$ with at most k roots. *In general* a medial therefore meets $\mathscr{R}$ in k separate points situated upon separate strata and separate generating lines. Hence the locus $\mathscr{R}$ is of *order* k. It is also of k dimensions, since it is the locus of ∞^1 $[k-1]$'s; so that it is classified as a V_k^k.

But it is also of *class* k, that is, a given medial in general position lies in exactly k primes which belong to $\mathscr{R}$. For, instead of taking a $2k \times k$ matrix, we may take its *transposed* $k \times 2k$ type

$$\Delta = [\Delta_1, \Delta_2] \tag{7}$$

with a pair of $k \times k$ components Δ_1 and Δ_2. If Δ is of rank k its k rows define k linearly independent primes and therefore a medial Δ common to all. This medial is

$$\text{env}\,[\omega\Delta_1, \omega\Delta_2], \tag{8}$$

where $\omega = [\omega_1, \ldots, \omega_k]$ is a set of parameters. Also the prime $[\omega\Delta_1, \omega\Delta_2]$ belongs to $\mathscr{R}$ if it is identical with the prime $[\sigma u, -\rho u]$: that is if

$$\omega(\rho\Delta_1 + \sigma\Delta_2) = 0 \quad (\omega \neq 0). \tag{9}$$

Hence $|\,\rho\Delta_1 + \sigma\Delta_2\,| = 0$, which in general has just k solutions for $\rho:\sigma$. Hence the scroll $\mathscr{R}$ is of class k.

Allowing for all possible ranks r, where $0 < r \leqslant k$, the matrix Δ represents a primary space, that is, a medial or else a higher dimensional space. If $r = 1$ it represents a prime. Thus the pencil of $k \times k$ matrices $D = \lambda D_1 + \mu D_2$ answers to the punctual, and its transposed form $\Delta = \lambda\Delta_1 + \mu\Delta_2$ to the primary, space. The scroll $\mathscr{R}$ and the basic medials A, C are the same for each.

Subordinate duality

6. It will be seen that the duality of points and primes in $[2k-1]$ induces a duality also within the space $[k-1]$ of each medial. Thus an arbitrary prime δ whose co-ordinates are $[u, v]$ meets the medial A in a $[k-2]$ given by $[u, 0]$, and meets C in another $[k-2]$ given by $[0, v]$. *Within* A this $[u, 0]$ is a *prime* of A, whose co-ordinates are u (a vector of k elements). Likewise v is a prime within C.

Just as a is used for a point of A, so α, β, etc., will denote primes $[2k-2]$ containing A, B, etc., respectively. Corresponding to the usual notation for A, B, C as strata of $\mathscr{R}$ we shall have the co-ordinates

$$[0, u], \quad [u, -u], \quad [u, 0], \tag{1}$$

for three such primes α, β, γ respectively which are *coaxal*, and therefore have a secundum G in common. Also this G which is a generating secundum of $\mathscr{R}$ meets each of A, B, C in a prime (within the medial) of parameter u. We may exhibit this feature by the reflexive arrow notation

$$\alpha \to \beta \to \gamma \quad \text{or briefly} \quad \alpha \to \gamma. \tag{2}$$

Also if, as above, Δ is any primary space contained by the prime δ, whose matrices Δ_1, Δ_2 are the transposed form of D_1, D_2, we can write

$$[0, \omega\Delta_2], \quad [\omega\Delta_1, \omega\Delta_2], \quad [\omega\Delta_1, 0], \tag{3}$$

for three primes α, δ, γ which are coaxal; and also

$$\alpha \twoheadrightarrow \delta \twoheadrightarrow \gamma \quad \text{or briefly} \quad \alpha \twoheadrightarrow \gamma, \tag{4}$$

where the double arrow has specific reference to this matrix Δ.

The single arrow will always refer to reflexion through B, while the double arrow always refers to D or Δ. The former is a (1, 1) correspondence: the latter is not necessarily so.

Grace's collineation defined by four medials

7. Take four medials A, B, C, D of $[2k-1]$ which are skew to one another. Let a, a', b, c, d be five points of them such that abc are in line, as also $a'cd$. That is

$$a \rightarrow b \rightarrow c \twoheadrightarrow d \twoheadrightarrow a'. \tag{1}$$

This sets up a (1, 1) correspondence between a and a', as both vary throughout their $[k-1]$ space A, for which there will be *in general* k latent points a_λ of A, where a and a' coincide, and the broken line $abcd$ becomes the straight $a_\lambda b_\lambda c_\lambda d_\lambda$.

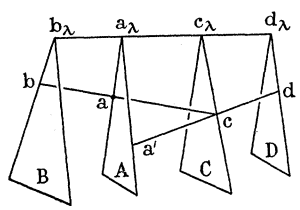

Figure 1

Hence there are exactly k line transversals of four medials in general, which will meet D in the k points d_λ, in fact where the scroll $\mathscr{R}$, defined by A, B, C, meets D. This gives an alternative proof that $\mathscr{R}$ is of order k.

Again the cross-ratios $(a_\lambda b_\lambda c_\lambda d_\lambda)$ on these k transversal lines must evidently be projective invariants of the four medials.

These ideas and results were given by Mr J. H. Grace (1929) in the course of his important account of 'Double Figures and Rational Normal Curves'.

If the co-ordinate vectors of a, a', b, c, d are x, ξ, y, z, t respectively, the analytical form of the collineation is obtained by eliminating y, z and t from the conditions of collinearity,

$$Ax+By+Cz = 0, \quad A\xi+Cz+Dt = 0. \tag{2}$$

With the canonical forms $(I, 0)$, (I, I), $(0, I)$ for three skew medials A, B, C and the general form (D_1, D_2) for D, we at once obtain

$$\{x, 0\} \rightarrow \{x, x\} \rightarrow \{0, x\} \twoheadrightarrow \{D_1\theta, D_2\theta\} \twoheadrightarrow \{D_1\theta, 0\}, \tag{3}$$

where necessarily $\rho x = D_2\theta$, in order to make the point c the same, whether obtained from ab or from $a'd$. The scalar factor ρ, which must be non-zero, can then be merged in the homogeneous co-ordinates x_i of x.

This gives Grace's collineation between x and ξ in the form

$$x = D_2\theta, \quad \xi = D_1\theta, \quad \theta = [\theta_1, \theta_2, \ldots, \theta_k], \tag{4}$$

in terms of the parameter θ. Incidentally this form is applicable for all such linear loci D, whether medial or of lower dimensions, and whether skew or otherwise. If, however, D is skew to A, then the matrix

$$[A, D] = \begin{bmatrix} I & D_1 \\ . & D_2 \end{bmatrix}$$

must be of full rank $2k$, so that its determinant $|D_2|$ is non-zero and D_2 is non-singular. Also θ can then be eliminated, and we have the explicit form

$$\xi = D_1 D_2^{-1} x = Hx, \quad H = D_1 D_2^{-1}. \tag{5}$$

Furthermore, by a change of the frame of reference within D we may absorb D_2 in the co-ordinate system and take $\{D_1 x, x\}$ for the typical point of D. The collineation within A is now given by $\xi = D_1 x$. Conversely, any collineation between two points a and a' of A is capable of the form (4), to which a definite space D belongs. This proves the following theorem:

THEOREM 2. *Any point-to-point collineation within a space* $[n]$ *can be constructed by Grace's method of transversals across any two further spaces* $[n]$ *and a suitably chosen fourth space of the same or lower dimensions, all four spaces being skew to each other but situated in* $[2n+1]$.

Again if D is skew to C, D_1 must be non-singular and

$$x = D_2 D_1^{-1}\xi = K\xi, \quad K = D_2 D_1^{-1}, \tag{6}$$

and, if D is skew to both A and C, then $K = H^{-1}$.

Again, if x is a latent point then $\mu x = Hx$, where μ is a latent root of H; and the collinear range of points, on A, B, C, D respectively, is now

$$\{x, 0\}, \quad \{x, x\}, \quad \{0, x\}, \quad \{\mu x, x\}.$$

But these four points have a cross-ratio μ, and this applies to each such range. Since any non-zero scalar multiple ρH of the matrix gives the same collineation of x and ξ in homogeneous co-ordinates, the cross-ratio for each different transversal will be equal to $\rho\mu$, where μ is the corresponding latent root. This proves the following theorem:

THEOREM 3. *Each transversal line of the spaces A, B, C, D meets A at a latent point of the collineation; and one of the six cross-ratios of the four points of intersection of the transversal line with A, B, C, D is proportional to the corresponding latent root.*

COROLLARY. *The same collineation can be interpreted as a one-to-one relation between the generating lines x, ξ of the scroll* $\mathscr{R}$. *The latent lines are those which meet the space D, and their cross-ratios are the corresponding latent ratios* (i.e. *ratios of latent roots*).

This corollary seems to provide the closest geometrical interpretation for the algebraic theory of a matrix pencil.

GROUP OF COLLINEATIONS DERIVED FROM FOUR MEDIALS

8. When A, B, C, D are four medials in general position, that is, skew to one another, both H and H^{-1} exist, and no latent root can vanish. There are clearly 24 collineations

obtainable by permuting the order of the four medials. Let (1234) denote the above case of a collineation from a to a' in A, hinging on C, and having a matrix H. Reference to the figure shows that (1432) would also hinge on C but would reverse the order, a' passing to a with a matrix H^{-1}.

Of the six permutations (1ijk) three move a to a new point in A, and three bring a new point back to a. It is straightforward to verify that the results are as follows:

collineation		matrix	latent root	
(1234)	a to a'	H	μ	
(1432)	a' to a	H^{-1}	μ^{-1}	
(1324)	a to a''	$I-H$	$1-\mu$	
(1423)	a'' to a	$(I-H)^{-1}$	$(1-\mu)^{-1}$	
(1243)	a to a'''	$H/(H-I)$	$\mu/(\mu-1)$	
(1342)	a''' to a	$I-H^{-1}$	$1-\mu^{-1}$	(1)

For example, in (1324) the collinearity conditions are

$$Ax+By+Cz = 0, \quad A\xi+By+Dt = 0, \tag{2}$$

which yield $\{x, 0\}$, $\{x, x\}$, $\{0, x\}$, $\{H\theta, \theta\}$, $\{\xi, 0\}$ for the five points a, b, c, d, a''. For bda'' to be in line, take $x = \theta$ and $\xi = (I-H)\,x$ on using (2), which agrees with the above table.

Each matrix is a scalar function of H, and thus provides a simple instance of Sylvester's theorem, that the latent root of $f(H)$ is $f(\mu)$ when that of H is μ. Of course μ takes each of the k or less values belonging to H in each of the above collineations.

Each of the six modes of the matrix gives rise to four collineations, one upon each medial. For example, the matrix H belongs to the set

$$(1234), \quad (2143), \quad (3412), \quad (4321).$$

All 24 collineations have the same set of transversals to determine their latent points and roots. The symbol ($ijkl$) for a collineation has been chosen to agree with that of the cross-ratio of any four collinear latent points, one on each medial.

The actual number of transversals depends on the character of H, and therefore on the position of D with reference to the medials A, B and C and their associated scroll $\mathscr{R}$.

Dual form of the collineation

9. If within A the point x lie on the $[k-2]$ space u, that is, a prime of A, then $ux = 0$. If also the point ξ lie on a prime v then $v\xi = 0$. Hence $(uD_2-vD_1)\,\theta = 0$, so that the collineation

$$x = D_2\theta, \quad \xi = D_1\theta, \tag{1}$$

induces a dual form of collineation

$$uD_2 = vD_1. \tag{2}$$

This means that, as x varies over the prime u, ξ varies over v. The primes u and v are therefore related by the original collineation in the manner (2). This form is applicable even when D_1 and D_2 are singular.

We may reach the same result by starting with five primes $\alpha, \beta, \gamma, \gamma', \delta$ containing the medials, A, B, C, C', D respectively, and such that

$$\gamma \rightarrow \beta \rightarrow \alpha \twoheadrightarrow \delta \twoheadrightarrow \gamma', \tag{3}$$

so that $\alpha\beta\gamma$ are coaxal and so are $\alpha\delta\gamma'$. This sets up a collineation between two such primes γ and γ' of C, say $[u, 0]$ and $[v, 0]$, which meet the medial A in two of its primes u and v. The whole relation between u and v is then

$$[u, 0] \rightarrow [u, -u] \rightarrow [0, u] \twoheadrightarrow [\omega\Delta_1, \omega\Delta_2] \twoheadrightarrow [v, 0], \tag{4}$$

which is satisfied when

$$u = -\omega\Delta_2, \quad v = \omega\Delta_1, \quad \omega = [\omega_1, \ldots, \omega_k]. \tag{5}$$

From this parametric form of the collineation between u and v we deduce the form

$$\Delta_2 x + \Delta_1 \xi = 0 \tag{6}$$

for the point collineation, by taking $ux = 0$, $v\xi = 0$ for all ω.

Since the same space D is now regarded both as a locus $\{D_1\theta, D_2\theta\}$ and an envelope $[\omega\Delta_1, \omega\Delta_2]$, we shall have

$$[\omega\Delta_1, \omega\Delta_2]\{D_1\theta, D_2\theta\} = 0$$

identically for all values of ω and θ: for this last condition merely states that every prime of the envelope must contain every point of the locus. Hence

$$\Delta_1 D_1 + \Delta_2 D_2 = 0, \tag{7}$$

a condition which holds universally whether either of Δ_i or D_i are singular or not. But

$$\left.\begin{aligned} &\text{if} \quad |D_1| \neq 0, \quad \text{then} \quad \Delta_1 = -\Delta_2 D_2 D_1^{-1} = -\Delta_2 K,\\ &\text{if} \quad |D_2| \neq 0, \quad \text{then} \quad \Delta_2 = -\Delta_1 H,\\ \text{also } &\text{if} \quad |\Delta_1| \neq 0, \quad \text{then} \quad H = -\Delta_1^{-1}\Delta_2,\\ &\text{if} \quad |\Delta_2| \neq 0, \quad \text{then} \quad K = -\Delta_2^{-1}\Delta_1. \end{aligned}\right\} \tag{8}$$

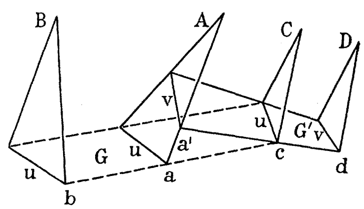

FIGURE 2

The figure illustrates the case for [5] when the medials A, B, C, D are planes. The prime α is a [4] containing the plane A and the [3] G (which have a line u in common). This G cuts B and C also in lines called u. The prime β contains B and G while γ contains C and G. These three lines u are the simplest visible sign of the relation $\alpha \rightarrow \beta \rightarrow \gamma$ between the coaxal primes. Simultaneously $\alpha \twoheadrightarrow \delta \twoheadrightarrow \gamma'$ gives rise to three lines v of A, u of C and v of D which lie in G', another [3].

When the line u varies in the plane A but always passes through a fixed point a, the corresponding lines u and v pass through related points b, c, d, a': and the connexion with the original figure is evident.

In general there are k distinct positions of u in A for which u and v coincide, and so do G and G'. They answer to the latent primes of the collineation and to coaxal sets α_μ, β_μ, γ_μ, δ_μ, and to a dual form of theorem 3.

Geometrical constructions of collineations

10. Before considering the bearing of this on matrix theory, let us notice a few particular consequences.

(1) Three pairs of points $x, \xi; x', \xi'; x'', \xi''$ upon a straight line A define a collineation on the line.

To construct it take any two further lines B and C in [3] which are skew to each other and to A. Then draw the transversal lines xyz, $x'y'z'$, $x''y''z''$, and thus obtain three lines ξz, $\xi' z'$, $\xi'' z''$. Any transversal D of the latter completes the figure from which any number of further pairs of points x, ξ of A can be found. Also the four lines A, B, C, D then have two transversals, the ratio of whose cross-ratios will be constant for all the ∞^1 positions of D, and will be equal to the latent ratios of the collineation. The ∞^1 of positions D will be a regulus of a quadric surface.

(2) In [5] four planes A, B, C, D in general position have three transversal lines.

(3) To construct any number of related pairs of points in a plane A having given four such pairs, and to find the latent ratios of their collineation, take two further planes B and C in [5], all mutually skew, and construct four lines ξz as before. Take any point t on one line, and the prime through t and two of the remaining three lines ξz. Such a prime will cut the fourth line in a point t'''. The two other of $\binom{3}{2}$ combinations yield points t', t'' on the second and third of the lines. All four points $tt't''t'''$ then lie simultaneously on three primes and therefore in one plane of [5], the plane D, which completes the collineation. Since t is arbitrary on the line ξz, D is one of ∞^1 such planes.

For all such D the three transversal lines of the planes A, B, C, D are cut in cross-ratios proportional to the three latent roots of the given collineation between x and ξ.

(4) Given $n+2$ pairs of corresponding points x, ξ in $[n]$, to find the collineation and its latent ratios, proceed similarly by constructing one of ∞^1 medials D from three skew medials in $[2n+1]$ derivable from the given pairs of points.

(5) The above theory is also that of two scrolls $\mathscr{R}$ and $\mathscr{R}'$ which have a distinct pair of strata A and C in common. One is defined by A, B, C, and the other by A, D, C.

(6) The scroll $\mathscr{R}$ is its own polar reciprocal with regard to each member of the quadric pencil $x'x = \lambda y'y$.

For this pencil is $\Sigma x_i^2 = \lambda \Sigma y_i^2$ and the result follows from the identity of the dual forms $x:y =$ constant and $u:v =$ constant, for the equations of $\mathscr{R}$.

Classification of a collineation

11. The general matrix pencil $\rho D_1 + \sigma D_2$ of k rows and columns can now be interpreted in terms of the scroll $\mathscr{R}$ (defined by the medials A, B, C) and the space D, this

latter being a $[k']$ with $k' < k$. Such a matrix pencil is known to be capable of the canonical form

$$\rho D_1 + \sigma D_2 = \text{diag}\,(L_l, M_m, N_n, O), \tag{1}$$

where L_l denotes a non-singular core, M_m a singular matrix of row dependence, N_n one of column dependence, and O a zero matrix. Various cases arise according to the presence or absence of one or other of these four parts.

First let L_l only occur. According to the theory of Weierstrass (1868) the non-singular core then consists of, say, h isolated *latent matrices*: that is

$$L_l = \text{diag}\,(L_{e_1}, L_{e_2}, \ldots, L_{e_h}), \tag{2}$$

where

$$L_{e_i} = \begin{bmatrix} \alpha\rho+\sigma & \rho & \cdot & \ldots & \cdot \\ \cdot & \alpha\rho+\sigma & \rho & \ldots & \cdot \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ \cdot & \cdot & \cdot & \ldots & \alpha\rho+\sigma \end{bmatrix}, \tag{3}$$

with e_i rows and columns, and with $\alpha\rho+\sigma$ occurring e_i times on the diagonal, ρ occurring once less on the over diagonal, and zeros elsewhere. This linear expression $\alpha\rho+\sigma$ is a factor of the characteristic determinant $|\rho D_1 + \sigma D_2|$ of the pencil. Furthermore, in this *non-singular* case, the determinant satisfies the relations

$$|\rho D_1 + \sigma D_2| = |L_l| = |L_{e_1}|\,|L_{e_2}| \ldots |L_{e_h}| \neq 0. \tag{4}$$

Also $e_1 + e_2 + \ldots + e_h = l = k$. Each L_{e_i} is a latent matrix, whose determinant $(\alpha\rho+\sigma)^{e_i}$ is an *elementary divisor* with an *exponent* e_i. Following Segre (1884 *b*, *c*) we characterize L_l by the expression

$$((e_1 e_2 \ldots)\,(e_f \ldots) \ldots (\ldots e_h)), \tag{5}$$

where exponents which correspond to the same value of α are grouped within a parenthesis. The ordinary case is given by $(11 \ldots 1)$, when there are n distinct values of α, and the scroll $\mathscr{R}$ is met by D at n points one on each of the n strata α.

From (3) it follows that the coefficient of σ in (2) is the unit matrix of order k, so that $|D_2| \neq 0$, and D must be a medial in this non-singular case.

Let the co-ordinates $\{x, y\}$ of a point of the scroll $\mathscr{R}$ be expressed as a two-rowed matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \lambda\phi_1, & \lambda\phi_2, & \ldots, & \lambda\phi_k \\ \mu\phi_1, & \mu\phi_2, & \ldots, & \mu\phi_k \end{bmatrix}, \tag{6}$$

where the ratio $\lambda : \mu$ fixes a stratum, and the k parameters ϕ_i fix a point of the stratum. In this notation the typical point $\{D_1\theta, D_2\theta\}$ of D appears as the two rowed matrix

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = [\Theta_{e_1}, \Theta_{e_2}, \ldots, \Theta_{e_h}], \tag{7}$$

where each of these h blocks is of the type

$$\Theta_{p+1} = \begin{bmatrix} \alpha\theta_0 + \theta_1, & \alpha\theta_1 + \theta_2, & \ldots, & \alpha\theta_{p-1} + \theta_p, & \alpha\theta_p \\ \theta_0, & \theta_1, & \ldots, & \theta_{p-1}, & \theta_p \end{bmatrix} \quad (p \geqslant 0). \tag{8}$$

In fact the first e_1 co-ordinates x_i, and also the first e_1 of the y_i, are obtained from the coefficients of ρ and σ respectively in $L_{e_1}\theta$, since $L_l = \rho D_1 + \sigma D_2$, where θ is a complete set of k parameters which includes $\theta_0, \ldots, \theta_{e_1-1}$. Similarly for each further Θ_{ei}, with e_i new parameters θ_j, until all the $\Sigma e_i = k$ parameters are exhausted. The verification is straightforward. The case $e_1 = 1$ has for its Θ a single column $\begin{bmatrix}\alpha\theta_0\\ \theta_0\end{bmatrix}$.

By identifying (6) and (7) we obtain all possible points common to D and $\mathscr{R}$. It follows that, within each set of e_i columns, corresponding to an elementary divisor and therefore to a Θ, there is exactly one common point, given by

$$\theta_0 \neq 0, \quad \theta_i = 0, \quad i > 0. \tag{9}$$

For the first elementary divisor we may take the point to be

$$s_1 = d_1 = \begin{bmatrix}\alpha, & 0, & \ldots, & 0\\ 1, & 0, & \ldots, & 0\end{bmatrix}, \tag{10}$$

namely, s_1 a point of the stratum S, which coincides with d_1 a point of D. Since the ratio $\alpha:1$ defines this stratum, the possibility of S being the stratum A is excluded, although it might happen to be C, with $\alpha = 0$.

In the *regular** case, when algebraically there is just *one invariant factor*, each e_i belongs to a distinct value of α, so that D meets $\mathscr{R}$ at h separate points one on each of h strata. In the irregular case, let exactly q of the e_i belong to the same α, so that exactly q of the initial parameters $\theta_0, \theta_0', \ldots$, from q of the blocks can simultaneously be non-zero. They will consequently furnish a linear $[q-1]$ space common to D and $\mathscr{R}$, given let us say by

$$S_{\mu_0} = D_{\mu_0} = \begin{bmatrix}\alpha\theta_0, & 0, & \ldots, & \alpha\theta_0', & 0, & \ldots, & \ldots\\ \theta_0, & 0, & \ldots, & \theta_0', & 0, & \ldots, & \ldots\end{bmatrix}, \tag{11}$$

which satisfies both (6) and (7). This space will lie in the stratum α, and D can only meet this stratum in this space.

It is now possible to describe the meeting of D and $\mathscr{R}$ in terms of the Segre characteristic (5), which is best written as a matrix of positive integers e_{ij},

$$\mathscr{E} = \begin{bmatrix}e_{11} & e_{12} & \cdots & e_{ij}\\ e_{21} & e_{22} & \cdots & \\ \vdots & & & \\ e_{q1} & \cdots & & \end{bmatrix}, \qquad \begin{matrix}\times & \times & \times & \times\\ \times & \times & \times & \\ \times & & & \end{matrix} \tag{12}$$

where the h integers $e_1, \ldots, e_h$ are now arranged in rows and columns, each column being associated with a particular value of α. Within each column the elements are arranged in descending value. The lengths of the rows and of the columns diminish (if necessary) from above to below and from left to right respectively, so that the non-zero portion of $\mathscr{E}$ forms a *tableau*, whose shape is rectangular or else is bounded on the right and below by a zigzag edge.

In this notation the matrix pencil has q invariant factors, one for each row of $\mathscr{E}$, each such factor consisting of the product of the distinct elementary divisors represented by the row. Collecting these results we have the theorem:

THEOREM 4. *When the matrix pencil $\rho D_1 + \sigma D_2$ is non-singular, then D is a medial which meets the scroll $\mathscr{R}$ on j different strata, where j is the number of distinct latent roots α, that is the*

* Segre, *Encyk.* p. 844. The word is due to Predella.

number of distinct linear factors $\alpha\rho+\sigma$ *of* $|\rho D_1+\sigma D_2|$. *Furthermore, within a stratum* $S(\alpha)$, D *meets* $\mathscr{R}$ *in a* $[q'-1]$, *where* q' *is the number of different elementary divisors (or invariant factors) associated with this latent root* α.

The particular form (3) above depends on the non-singularity of D_2, that is, on the failure of D to meet the stratum A. Were D to meet A without $|\rho D_1+\sigma D_2|$ vanishing identically, a change of basis to a new A, distinct from C and not meeting D, could be made before reducing the pencil to the above canonical form. The condition (4) is, however, characteristic of the non-singular case, and the above integer j is essentially finite.

Multiple contact of D with $\mathscr{R}$

12. When any exponent e_i exceeds unity several intersections of D with $\mathscr{R}$ have evidently coincided, and D may be said to have p-fold contact with $\mathscr{R}$, where $p = e_i-1$. This may be interpreted geometrically as follows:

Let a_i, b_i, c_i, s_i denote those points of the strata A, B, C, S which lie on the generating line g_i (as given by the vanishing of all ϕ except ϕ_i in § 11 (6)). Also $\mu = 0$ for a_i, $\lambda = 0$ for c_i, $\lambda = \mu$ for b_i and $\lambda:\mu = \alpha:1$ for s_i.

Further, let $A_{ijk\ldots}$ denote the space defined by, and containing, the independent points a_i, a_j, a_k, ... Similarly for the other spaces. Finally, let d_i denote that point of D for which all θ except θ_{i-1} vanish in § 11 (7). When θ_0 alone is non-zero there is a point d_1 coinciding with s_1; but when θ_0, θ_1 alone are non-zero we obtain a line, say,

$$d_1 d_2 = D_{12} = \begin{bmatrix} \alpha\theta_0+\theta_1, & \alpha\theta_1, & 0, & \ldots, & 0 \\ \theta_0, & \theta_1, & 0, & \ldots, & 0 \end{bmatrix}. \tag{1}$$

This is the sum of two matrices, each of k columns,

$$S_{12} = \begin{bmatrix} \alpha\theta_0, & \alpha\theta_1, & 0, & \ldots, & 0 \\ \theta_0, & \theta_1, & 0, & \ldots, & 0 \end{bmatrix}, \qquad a_1 = \begin{bmatrix} \theta_1, & 0, & \ldots, & 0 \\ 0, & 0, & \ldots, & 0 \end{bmatrix},$$

from which we infer at once that the point a_1 of A is in line with any point s of the *line* S_{12} and d of the line D_{12}. Conversely, since θ_2, θ_3, ... all vanish, the only points d and s of D and S respectively, which are in line with a_1 of A must lie on the lines D_{12} and S_{12}. Hence (§ 2 (9)) the point a_1 is reflected through D to the *line* S_{12}, namely,

$$a_1 \twoheadrightarrow D \twoheadrightarrow S_{12}.$$

Again, when all θ except θ_0, θ_1, θ_2 vanish we should obtain a plane D_{123}, a plane S_{123}, and a line A_{12}, such that any point a of A_{12} is in line with d and s, if and only if d is on D_{123} and s on S_{123}. And so on.

Now suppose that D meets $\mathscr{R}$ at a point d_1 belonging to the elementary divisor L_e ($e = p+1>1$). Then d_1 is s_1 a point of the stratum S which may be reflected along a generator g_1 through B to a_1 of A, and thence through D to S, and thence back through B to A, and so on alternately. The result will be

$$d_1 = s_1 \rightarrow a_1 \twoheadrightarrow S_{12} \rightarrow A_{12} \twoheadrightarrow S_{123} \rightarrow A_{123} \ldots \twoheadrightarrow S_{e!} \rightarrow A_{e!},$$

where $S_{e!} \equiv S_{123\ldots e}$. The process will terminate at the pth stage since the parameter θ_{p+1} does not exist in the matrix Θ_p to provide a further point s_{p+1}, so that the process would thereafter merely repeat the eth spaces of S and A identically.

Hence the given point d_1 common to D and $\mathscr{R}$ defines uniquely in A a point a_1, a line through it, A_{12}, a plane through the line, A_{123}, and so on until a $[p-1]$ is reached. There is a corresponding nest of spaces within each stratum, including S, and further a unique set

$$(D)_e \equiv d_1,\ D_{12},\ D_{123},\ \ldots,\ D_{e!},$$

of such spaces in D. The $p+1$ successive points $d_1, d_2, \ldots, d_e$ which define this set may be called a *chain* of length p. Since each such point depends on one new parameter θ_i, when the choice of the first i such points has been fixed, the next point lies on a fixed line within its space $D_{(i-1)!}$.

We now have the following result:

THEOREM 5. *When the matrix pencil is non-singular, each meeting point of D with $\mathscr{R}$ other than an ordinary intersection ($e = 1$) sets up a chain of points with a length $p = e-1$. This chain is obtained by successive reflexion of the initial point in B and D alternately, and is one of ∞^p such chains starting at the same point and lying in a fixed nest of spaces $(D)_e$. The initial point is then a point of p fold contact between D and $\mathscr{R}$.*

COROLLARY. *In the regular case, when the non-singular matrix pencil has a single invariant factor, D meets $\mathscr{R}$ at isolated points on distinct strata, each of which sets up a separate chain.*

THE CASE OF SEVERAL INVARIANT FACTORS

13. This case can be investigated by the same methods which are best explained by a typical example. We assume D to have a characteristic $\mathscr{E}$ with j columns, so that D meets the scroll $\mathscr{R}$ on exactly j different strata. Let it meet one such stratum S in a $[q'-1]$, so that q' of the elementary divisors belong to S. Each point common to D and S will have a chain as before, but these chains will not necessarily have the same length, and it remains to examine them.

For example, let exactly three indices of $\mathscr{E}$ belong to the root α, and let the corresponding part of D be given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \alpha\theta_0+\theta_1, & \alpha\theta_1+\theta_2, & \alpha\theta_2+\theta_3, & \alpha\theta_3+\theta_4, & \alpha\theta_4, & \alpha\theta_0'+\theta_1', & \alpha\theta_1', & \alpha\theta_0'' \\ \theta_0, & \theta_1, & \theta_2, & \theta_3, & \theta_4, & \theta_0', & \theta_1', & \theta_0'' \end{bmatrix}$$
$$= [\Theta,\ \Theta',\ \Theta'']. \tag{1}$$

This part of D is a $[7]$, say $D_{12\ldots8}$, where d_i denotes the ith point and answers to the ith among the eight parameters $\theta_0, \ldots, \theta_0''$. (Thus d_3 is given by all except θ_2 vanishing.) This $D_{8!}$ meets $\mathscr{R}$ in a plane S_{168}, say, which belongs to the stratum S. Points of this plane are given parametrically by $\theta_0, \theta_0', \theta_0''$ non-zero, the rest zero.

By a *nest* of spaces in general is meant a set $\Sigma, \Sigma_1, \Sigma_2, \ldots$ where Σ contains Σ_1, Σ_1 contains Σ_2, etc. It is useful to note that if two such nests overlap, their common parts also form a nest. For if PQ, QR are the initial spaces of the nests, with a common space Q, the common space Q_1 of the second members of the nests must necessarily belong to Q: and so on.

Now reflect S_{168} through B to A. This gives A_{168}. Inspection of (1) shows that the only points of A_{168} which are in line with points of D and of S are those where θ_1, θ_1' are non-zero, but $\theta_2 = \theta_3 = \theta_4 = 0$. This means that reflexion of A_{168} through

D to S produces the space S_{12678}, obtained in fact by retaining those parameters whose suffixes are 0 or 1 only. Thus

$$A_{168} \twoheadrightarrow D \twoheadrightarrow S_{12678}.$$

When such alternate reflexion in B and D is continued as before, we obtain the following result:

$$D_{168} = S_{168} \rightarrow A_{168} \twoheadrightarrow S_{12678} \rightarrow A_{12678} \twoheadrightarrow S_{123678} \rightarrow A_{123678} \twoheadrightarrow S_{1234678} \rightarrow A_{1234678} \twoheadrightarrow S_{8!} \rightarrow A_{8!}. \tag{2}$$

Thereafter the sequence is stable. These terms are derived as in § 12 by adjoining a new non-zero set θ_i, θ'_i, ... at each double-arrow stage $(i = 1, 2, \ldots)$ until the parameters are exhausted. The process therefore determines the nest of spaces A_{168}, A_{12678}, A_{123678}, ..., but not the precise positions of the points $a_1, \ldots, a_8$.

Now consider the tableau

$$\begin{matrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & & & & \end{matrix} \tag{3}$$

which is constructed columnwise by placing 3 marks in the first column, 5 marks in the first two, 6 in the first three, 7 in the first four and 8 in the first five, according to the numbers of suffixes in successive A's. As a result the numbers of marks in the rows are 5, 2, 1, which are the indices of the elementary divisors, as shown by the structure of the matrix (1) with its five, two, and single column blocks Θ, Θ', Θ''.

But this feature is true in general and gives the values of the q indices e for the elementary divisors belonging to one stratum, by counting the marks on the rows of the tableau. For the shortening of the columns is due to the falling out of the second-term parameters in the top row of (1), so that, for instance, the third mark in the second column fails since no θ''_1 is present, and the absence of θ'_2 shortens column 3.

In general therefore D meets a stratum S of $\mathscr{R}$ in a $[q-1]$, let us say S_{μ_0}. By alternate reflexions in B and D respectively this gives

$$S_{\mu_0} \rightarrow A_{\mu_0} \twoheadrightarrow S_{\mu_1} \rightarrow A_{\mu_1} \twoheadrightarrow S_{\mu_2} \rightarrow \ldots, \tag{4}$$

where either set (A_μ) and (S_μ) is a nest of spaces each of which is contained by its successor. From the dimensions of either, say, $\nu_1 - 1$, $\nu_2 - 1$, ..., we form the first differences $\epsilon_1, \epsilon_2, \ldots$; that is

$$\epsilon_1 = \nu_1, \quad \epsilon_2 = \nu_2 - \nu_1, \quad \ldots, \quad \epsilon_i = \nu_i - \nu_{i-1}, \quad \ldots. \tag{5}$$

The manner of generation shows that they satisfy the relations

$$\epsilon_1 \geqslant \epsilon_2 \geqslant \epsilon_3 \geqslant \ldots. \tag{6}$$

We then construct a tableau whose *columns* are of lengths $\epsilon_1, \epsilon_2, \ldots$, which represent a partition $\{\epsilon_1 \epsilon_2 \ldots\}$ of the positive integer $\pi = \Sigma\epsilon$ (or ν_q). The same tableau gives what is called the *conjugate partition* $\{\epsilon_1 \epsilon_2 \ldots\}'$ by means of its *rows*. Hence the partitional relation

$$\{\epsilon_1 \epsilon_2 \ldots\}' = \{e_1 e_2 \ldots e_q\} \tag{7}$$

gives the indices e_i of the elementary divisors corresponding to the root α.

For example, $\{32111\}' = \{521\}$ for a pair of conjugate partitions of $\pi = 8$ in (3).

THEOREM 6. *The total meeting S_{μ_0} of D with the stratum S of the scroll $\mathscr{R}$, in the non-singular case, gives rise, by successive reflexion in B and D alternately, to a nest of spaces (A_μ) in any other stratum A. The first differences, of the dimensions increased by unity, of successive members of the* *nest constitute a partition of a positive integer π, whose conjugate partition yields the set of indices e of the elementary divisors belonging to the said stratum.*

Polar chains of points

14. While the above method of reflexion gives the indices of the elementary divisors it does not give the precise distribution of corresponding intensities of contact between D and $\mathscr{R}$. Each point d of the $[q'-1]$ common to D and the stratum S will initiate a chain whose index is one or other of the indices e belonging to the corresponding column of $\mathscr{E}$. It remains to find them.

To do this we define a certain space $[k]$ as the *tangent supermedial* (T.S.M.) of a point ϕ upon $\mathscr{R}$, namely, one given by the $k-1$ linear equations

$$T_\phi: \frac{\mu x_1 - \lambda y_1}{\phi_1} = \ldots = \frac{\mu x_k - \lambda y_k}{\phi_k}. \tag{1}$$

Here x, y are current co-ordinates, and the ratio $\lambda:\mu$ with the ϕ_i give that point $\{\lambda\phi, \mu\phi\}$ of $\mathscr{R}$ to which the T.S.M. belongs. It is readily seen to be the space of dimension one higher than that of a medial, and it includes and is determined by the stratum S and generating line g which intersect at the point ϕ of $\mathscr{R}$. It is also the common part of all those tangent primes of the quadrics $x_i y_j = x_j y_i$ which have a point of contact at this point ϕ.

By the *polar supermedial* (P.S.M.) of a point d, which need not lie on $\mathscr{R}$, is meant the T.S.M. of that point s, belonging to the stratum S, which is obtained by reflecting d through A to a point c of C, and c again through B to s of S. That is $a \twoheadrightarrow d \twoheadrightarrow c \rightarrow B \rightarrow s$, or

$$\{D_1\theta, 0\} \twoheadrightarrow \{D_1\theta, D_2\theta\} \twoheadrightarrow \{0, D_2\theta\} \rightarrow B \rightarrow \{\lambda D_2\theta, \mu D_2\theta\} = s. \tag{2}$$

Hence the equations (1) give the polar of $d = \{D_1\theta, D_2\theta\}$ by taking $D_2\theta = \phi$. Such a polar is defined with respect to $\mathscr{R}$ and S. When d is on $\mathscr{R}$ the polar becomes the tangent supermedial.

The chain of points $d_1 d_2 \ldots d_e$ of Theorem 5 can now be generated in an alternative way according to the following theorem:

Theorem 7. *When D touches $\mathscr{R}$ at an isolated point $d_1 = s_1$ of a stratum S, then T_1 the tangent supermedial of d_1 will meet D in a line D_{12} through d_1: also P_2 the polar of any new point d_2 on this line will meet D in a line D_{13} through d_1: and P_3 the polar of any new point d_3 of D_{13} will meet D in a line D_{14} through d_1; and so on, until a point d_e is reached, in $p = e-1$ steps, whose polar meets D in the point d_1 only.*

Such a *polar chain*, of length p, is one of ∞^p chains of equal length initiated by the point d_1 of index e: and it is identical with the chain of theorem 5. The points d_i also determine the same nest of spaces $d_1, \ldots, D_{e1}$ as before.

Proof. Take $\lambda = \alpha$, $\mu = 1$ in (2), and let $d = \{D_1\theta, D_2\theta\}$, any point of D, be denoted briefly by $\{\theta\}$. Consider then the following sequence of points of D,

$$\left.\begin{array}{ll} d_1 = \{\xi_0, 0, \ldots \quad\quad \ldots, 0\}, & \xi_0 \neq 0, \\ d_2 = \{\xi_1, \xi_0, 0, \ldots \quad \ldots, 0\}, & \xi_1 \text{ arbitrary}, \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ d_e = \{\xi_p, \xi_{p-1}, \ldots, \xi_1, \xi_0, 0, \ldots, 0\}, & \xi_p \text{ arbitrary}, \end{array}\right\} \tag{3}$$

where the θ successively take these particular values ξ_i or zero. When D touches $\mathscr{R}$ at an isolated point we may take the corresponding Θ of § 11 (8), and d_1 to be the point of contact. For this point we have $\phi = D_2\theta = \{\xi_0, 0, \ldots, 0\}$, so that (1) gives the tangent supermedial of d_1 as

$$T_1;\ x_i - \alpha y_i = 0 \quad (i = 2, 3, \ldots, k).$$

On substituting in T_1 the parametric values of the x_i and y_i as given by

$$\Theta = \begin{bmatrix} \alpha\theta_0 + \theta_1, & \ldots, & \alpha\theta_p \\ \theta_0, & \ldots, & \theta_p \end{bmatrix}, \quad x_i = y_i = 0 \quad (i = p+2, \ldots, k),$$

we find that all θ_i must vanish for $i > 1$. Hence D meets T_1 in a *line* $\{\theta_0, \theta_1, 0, \ldots, 0\}$ with arbitrary θ_0, θ_1. Any new point of this line D_{12} other than d_1 is therefore d_2 as shown above.

Having chosen d_2 and therefore ξ_0, ξ_1, we similarly find P_2 the polar of d_2. It is

$$P_2;\ \frac{x_1 - \alpha y_1}{\xi_1} = \frac{x_2 - \alpha y_2}{\xi_0}, \quad x_i - \alpha y_i = 0 \quad (i = 3, \ldots, k), \tag{4}$$

and this meets D only where $\theta_1/\xi_1 = \theta_2/\xi_0$ and all θ_i vanish for $i > 2$, as is again seen by substitution. Such a point of meeting is the above d_3 with ξ_2 arbitrary. It must therefore lie on a line D_{13} through d_1, and so on.

But at the stage P_{p+1} the corresponding substitution gives

$$\frac{\theta_1}{\xi_p} = \frac{\theta_2}{\xi_{p-1}} = \ldots = \frac{\theta_{p-1}}{\xi_1} = \frac{0}{\xi_0}, \tag{5}$$

so that D meets the polar where all θ_i except θ_0 vanish, that is at the point d_1 only. Q.E.D.

The space $D_{e!}$ is then the locus of d_e for all its ξ_i arbitrary; similarly for its subspaces. This gives the nest $(D)_e$ of theorem 5.

THEOREM 8. *When D meets the stratum S in an $[\epsilon_1 - 1]$ space D', then the* T.S.M. *of δ_1, any point of D', will meet D in an $[\epsilon_1]$ space D'' containing D; and the polar of any point δ_2 of D'' outside D' will meet D in an $[\epsilon_1]$: and so on, until the stage when the new space $D^{(e)}$ coincides with D'. Thus each point of D' has a chain of a certain length $p = e-1$; and this index e will be one or other of the e_i associated with this stratum.*

Proof. Consider the points

$$\left.\begin{aligned} \delta_1 &= \{\xi_0,\ 0,\ \ldots,\ 0,\ \xi_0',\ 0,\ \ldots,\ 0,\ \xi_0'',\ 0,\ \ldots\}, \\ \delta_2 &= \{\xi_1,\ \xi_0,\ \ldots,\ 0,\ \xi_1',\ \xi_0',\ \ldots,\ 0,\ \xi_1'',\ \xi_0'',\ \ldots\}, \end{aligned}\ \text{etc.}\right\} \tag{6}$$

defined analogously to those of (3), but with one set ξ or ξ' or ξ'' for each block Θ, Θ', Θ'', of D belonging to the same root α. We assume e_1 columns of (6) to belong to ξ, e_2 to ξ', e_3 to ξ'', etc., where $e_1 \geq e_2 \geq e_3$.

As in (3) each point δ_i is here stated by giving the k parameters

$$\theta = \{\theta_0, \theta_1, \ldots, \theta_{e_1}, \theta_0', \ldots, \theta_{e_2}', \theta_0'', \ldots\}$$

particular values ξ_r or zero, as shown. Proceeding as before, we find that the P.S.M. of any point ϕ will meet D at a point, of parameter θ, such that

$$\phi_1:\phi_2:\ldots:\phi_k = \theta_1:\theta_2:\ldots:\theta_{e_1-1}:0:\theta_1':\ldots:\theta_{e_2-1}':0:\theta_1'':\ldots. \quad (7)$$

Hence, by taking the ϕ_i to be the ξ_i of δ_1, we find that δ_2 is the locus of intersection of T_1 and D, with the ξ_1, ξ_1', ξ_1'' of the leading columns arbitrary. Similarly for δ_3 from δ_2, with ξ_2, ξ_2', ξ_2'' arbitrary.

As in (5) above, the process alters as soon as a $\xi_0^{(i)}$ reaches the end of its block. Since $e_1 \geqslant e_2 \geqslant e_3$ this happens first for ξ_0'' at the point δ_{e_3}. If therefore $\xi_0'' \neq 0$, *all* earlier θ's of (7) would vanish for the next successive δ. Since $\theta_0, \theta_0', \theta_0''$ are absent from (7), they, and they alone, are arbitrary in the new δ; that is, $\delta_{e_3+1} = \delta_1$.

Also the locus of δ_1 is an $[\epsilon_1 - 1]$ where ϵ_1 is the number (three in the illustration (6)) of blocks Θ, while the locus of each subsequent δ_i has the same number of arbitrary constants with a set of non-zero fixed constants. This gives an $[\epsilon_1]$. Q.E.D.

Allowing for repetitions of indices let $\epsilon, \epsilon', \epsilon'', \ldots$ and $e, e', e'', \ldots$, denote the *distinct* values of the ϵ_i and e_i in the partitions (§ 13 (7)) arranged in descending order. The tableau, for example,

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 6 & 7 & & & \\ 8 & & & & \end{matrix} \qquad \begin{matrix} \times & \times & \times & \times & \times \\ \times & \times & & & \\ \times & & & & \end{matrix} \qquad (8)$$

will now give the partitions $\{ee'e''\} = \{521\}$ and $\{\epsilon\epsilon'\epsilon''^3\} = \{321^3\}$ by rows and columns.

By taking all the ξ of zero suffix, in δ_1, and allowing them to vanish one by one, starting from the right we obtain the following result:

COROLLARY. *The space D' common to D and S can be further subdivided into a nest of spaces $[\epsilon - 1]$, $[\epsilon' - 1]$, ..., starting with $D' = [\epsilon_1 - 1]$. Points of the innermost space will have index e, further points of the next containing space, index e', and so on, until the remaining points of the largest space D' have the smallest index e_i.*

The argument is made clear by the tableau. The above (8) when expressed in the form (6) gives

$$\delta_1 = \{\xi_0, \cdot, \cdot, \cdot, \cdot, \xi_0', \cdot, \xi_0''\}, \quad \operatorname{loc}\delta_1 = D_{168}.$$

If $\xi_0'' \neq 0$, the theorem shows that no further δ_i exists. Therefore D meets S in a *plane* $D' \equiv D_{168}$, and any point of this plane except points of the line D_{16} ($\xi_0'' = 0$) has unit index and no chain; the T.S.M. of the point meets D in D' only.

Next if $\xi_0'' = 0$, then $\operatorname{loc}\delta_1 = D_{16}$ (given by ξ_0, ξ_0'), and δ_2 exists. If also $\xi_0' \neq 0$, then the chain ends at δ_2. Any point of D_{16} except d_1 ($\xi_0' = 0$) therefore has index 2.

Lastly, if $\xi_0'' = \xi_0' = 0$, $\xi_0 \neq 0$, δ_1 is the point d_1 only, and the chain can reach δ_5; that is *one* point d_1 of the *line* D_{16} in the *plane* D_{168} has index 5. The nest of the corollary consists of this plane, line and point, whose dimensions are determined by the three different lengths $\epsilon, \epsilon', \epsilon''$ of column in the tableau.

The case when several e_i are equal, so that the tableau has several equal rows, is easily analysed. Thus if exactly ϵ_k of the longest rows are equal then a certain $[\epsilon_k - 1]$ of D' would have points of highest index e_1; similarly for the lower rows. The nest of spaces in fact characterizes the increasing intensities of contact between D and $\mathscr{R}$.

PHILOSOPHICAL TRANSACTIONS OF THE ROYAL SOCIETY A MATHEMATICAL, PHYSICAL & ENGINEERING SCIENCES

Rational normal loci

15. The nest of spaces formed by polar chains originating in a region D_{μ_0} common to D and the scroll may be regarded as an *osculating system* of a certain rational normal locus $\mathscr{N}$ contained by the scroll. For example, the point, line, plane, ..., $[p]$ of such a nest originating in a single point of contact d_1 of index $e = p+1$ prove to be the point of contact, tangent line, osculating plane, ..., osculating $[p]$ of a rational normal curve Γ of order e. This curve which lies on $\mathscr{R}$ is contained by that space $[e]$ which is defined by the $[p]$ and the external point a_e of A.

In fact, the points $[x', y']$ given by the $e+1$ successive rows of

$$\begin{bmatrix} \alpha & . & . & \cdots & . & \cdots & . & 1 & . & . & \cdots & . \\ 1 & \alpha & . & & . & & . & . & 1 & . & & . \\ & 1 & \alpha & & . & & . & . & . & 1 & & . \\ \cdots & \cdots & \cdots & \cdots & \cdots & & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots & \alpha & & \cdots & \cdots & \cdots & \cdots & \cdots & 1 \\ . & . & . & & 1 & & . & . & . & . & & . \end{bmatrix} \tag{1}$$

form such an osculating system for the curve

$$\Gamma : \mathrm{loc}\,\{\alpha+v,\ \alpha v+v^2,\ \ldots,\ \alpha v^p+\alpha v^{p+1},\ 0,\ \ldots,\ 0;\quad 1,\ v,\ \ldots,\ v^p,\ 0,\ \ldots,\ 0\}.$$

This is a rational normal curve of order e with parameter v and fixed α. The values of Γ, $\partial\Gamma/\partial v$, $\frac{1}{2}\partial^2\Gamma/\partial v^2$, etc., at $v = 0$ give the points represented by the successive rows of (1). But the first row denotes the point d_1: hence the first two give the tangent line, the first three the osculating plane, at d_1; and so on until the final row gives the point a_e, so that all $e+1$ rows give the space $[e]$ containing Γ.

Hence in the *regular* case there is one such curve for each distinct latent root α at an isolated contact of D and $\mathscr{R}$. The curve meets each generating line and stratum in one point only.

When two invariant factors exist D meets a stratum S on a straight line, and this gives rise to a *ruled surface* $\mathscr{N}$ upon the scroll. For example, in the case $\begin{smallmatrix} \times & \times & \times \\ \times & \times & \end{smallmatrix}$, $\begin{smallmatrix} 1 & 2 & 3 \\ 4 & 5 & \end{smallmatrix}$, D is given by

$$\{\alpha\theta_0+\theta_1,\ \alpha\theta_1+\theta_2,\ \alpha\theta_2,\ \alpha\theta_0'+\theta_1',\ \alpha\theta_1',\ \theta_0,\ \theta_1,\ \theta_2,\ \theta_0',\ \theta_1'\},$$

which meets $\mathscr{R}$ in a line $d_1 d_4$ consisting of points of index 2 and one exceptional point of index 3. Corresponding to this is the surface

$$\mathscr{N} : \mathrm{loc}\,\{\alpha u+uv,\ \alpha uv+uv^2,\ \alpha uv^2+uv^3,\ \alpha w+wv,\ \alpha wv+wv^2,\ u,\ uv,\ uv^2,\ w,\ wv\}, \tag{2}$$

where v and the ratio $u:w$ are the two parameters of the surface, while α is fixed. This surface evidently lies on $\mathscr{R}$ since all $x_i : y_i = \alpha+v$. Thus it meets each stratum where $v =$ constant, in a line, since $\mathscr{N}$ is linear in u and w. Also it contains ∞^1 normal curves which are cubics ($u:w =$ constant) and one conic ($u = 0$). Each of these *directrix* curves meets each generator (v) of the surface in one point. As in (1), the points given by $\mathscr{N}$ and its successive derivatives with regard to v at $v = 0$, for a fixed ratio $u:w$, again form a polar chain which defines an osculating system for the directrix curve through the initial point d ($u:w =$ constant). The order of the curve is the index of the initial point.

There is one such surface $\mathcal{N}$ for each distinct latent root α, whenever there are two invariant factors of D.

For the case of three invariant factors a twisted solid $\mathcal{N}$ is taken, consisting of ∞^2 directrix curves Γ lying on $\mathcal{R}$ and meeting each stratum in a plane; and for q invariant factors ∞^{q-1} such curves meeting a $[q-1]$. The construction of $\mathcal{N}$ for these cases is obvious. Perhaps the most interesting geometrical case is when all the indices are equal.

For example, for the characteristic $((222))$

$$\mathcal{N}: \text{loc}\,\{\alpha u+uv,\ \alpha uv+uv^2,\ \ldots,\ \alpha u''v+u''v^2,\ u,\ uv,\ u',\ u'v,\ u'',\ u''v\},$$

which is a surface of ∞^2 conics in $[11]$ each meeting the plane $d_1 d_3 d_5 \equiv D_{135}$ in one point. The tangents to the conics at these points, which fill the plane, relate it to another plane D_{246}, each tangent constituting a chain of length 1 (and index 2). Such tangents form a stratified locus such as $\mathcal{R}$ but in the subspace $[5]$, that is in the medial D.

The singular case

16. We now suppose that the matrix pencil $\rho D_1+\sigma D_2 = \text{diag}\,(L_l, M_m, N_n)$ involves the singular matrices M or N or both. More expressly (Turnbull & Aitken 1932) we take

$$M_m = \text{diag}\,(M_{m_1}, \ldots, M_{m_\mu}), \quad N_n = \text{diag}\,(N_{n_1}, \ldots, N_{n_\nu}), \tag{1}$$

where each submatrix is of one or other type

$$M_{m_i} = \begin{bmatrix} \rho & & & \\ \sigma & \rho & & \\ & \sigma & & \\ & & \ddots & \\ & & & \rho \\ & & & \sigma \end{bmatrix}, \quad N_{n_i} = \begin{bmatrix} \rho & \sigma & & & \\ & \rho & \sigma & & \\ & & \ddots & & \\ & & & \rho & \sigma \end{bmatrix}, \tag{2}$$

with m_i columns and m_i+1 rows, and n_i+1 columns and n_i rows respectively. Furthermore, M_m must have m columns and $m+\mu$ rows, while N must have $n+\nu$ columns and n rows. Evidently

$$m = m_1+m_2+\ldots+m_\mu, \quad n = n_1+n_2+\ldots+n_\nu, \tag{3}$$

so that the whole matrix $\rho D_1+\sigma D_2$ in its canonical form has $l+m+n+\nu$ non-zero and independent columns, that is a column-rank r, where

$$r = l+m+n+\nu \leqslant k, \tag{4}$$

while the row-rank r' is given similarly by

$$r' = l+m+n+\mu \leqslant k. \tag{5}$$

Hence also D is a punctual space $[k']$, where $k' = r-1$, or else, as we shall see in § 20, is a primary space $[k'']$, where $k'' = r'-1$.

Proceeding as before we now find that the co-ordinates $\{x, y\}$ of a point of D are given by three types of matrix Θ, Φ, Ψ, corresponding to the L, M, N; namely,

$$\text{loc}\,D = \begin{bmatrix} x' \\ y' \end{bmatrix} = [\Theta_1, \ldots, \Theta_h, \Phi_1, \ldots, \Phi_\mu, \Psi_1, \ldots, \Psi_\nu], \tag{6}$$

where Θ is as before. From (2) we find that

$$\Phi_i = \begin{bmatrix} \phi_1 & \phi_2 & \phi_3 & \dots & \phi_{m_i} & \cdot \\ \cdot & \phi_1 & \phi_2 & \dots & \phi_{m_i-1} & \phi_{m_i} \end{bmatrix}, \quad \Psi_i = \begin{bmatrix} \psi_0 & \psi_1 & \dots & \psi_{n_i-1} \\ \psi_1 & \psi_2 & \dots & \psi_{n_i} \end{bmatrix}, \tag{7}$$

where the leading zero of the lower line and the final zero of the upper line in each Φ_i is essential. In all these $h+\mu+\nu$ submatrices the sets of parameters are entirely distinct. Each Θ_i has its e_i parameters, giving a total of l for the non-singular core: each Φ_i has m_i, and each Ψ_i has n_i+1: in all m and $n+\nu$ respectively by (3). The k parameters of θ in the usual $\{D_1\theta, D_2\theta\}$ are these $l+m+n+\nu = r$ parameters, in this order, followed if necessary by $k-r$ zeros.

The points common to D and $\mathscr{R}$ can be found as before, by equating (6) to $\{\rho\theta, \sigma\theta\}$, piece by piece: that is, the upper and lower rows of (7) must be proportionals. This at once gives all ϕ_i zero, so that *no point d of Φ can lie on $\mathscr{R}$*. Also for each Ψ the co-ordinates must be in geometrical progression, and the parameters corresponding to Ψ_i must be of the form

$$\psi = \{\rho^p, \rho^{p-1}\sigma, \rho^{p-2}\sigma^2, \dots, \sigma^p\} \quad (p = n_i), \tag{8}$$

to a constant numerical factor. Hence each space Ψ_i of D meets $\mathscr{R}$ in a certain rational normal curve $\mathscr{N}_i$ of order n_i, and this curve meets each stratum $(\rho:\sigma)$ once and once only.

If $\nu = 2$, then D contains Ψ_1 and Ψ_2, so that it will meet $\mathscr{R}$ in a rational normal surface $\mathscr{N}$, whose typical point is

$$\psi = \{u\rho^p, u\rho^{p-1}\sigma, \dots, u\sigma^p, w\rho^q, \dots, w\sigma^q\} \quad (p = n_1,\ q = n_2), \tag{9}$$

where the terms in u correspond to Ψ_1, and those in w to Ψ_2. For this is the most general solution for Ψ and $\mathscr{R}$; similarly for higher values of ν. Such a locus $\mathscr{N}$ will meet each stratum of $\mathscr{R}$ in a $[\nu-1]$; it is of the type already noticed in connexion with osculating systems.

On combining each Θ which belongs to a stratum $S(\alpha)$ with the whole set Ψ, we obtain the complete meeting of D with S. It will now be an $[\epsilon_1+\nu-1]$, which contains the space D', an $[\epsilon_1-1]$, peculiar to S as before, together with the $[\nu-1]$ in which $\mathscr{N}$ meets S. This D' is no longer unique within S, as it was in the non-singular case, but may be any portion of the larger space $[\epsilon_1+\nu-1]$ provided that it is entirely distinct from $\mathscr{N}$. Hence

THEOREM 9. *If, and only if, D meets each stratum of $\mathscr{R}$, then the matrix pencil possesses a singularity N of column dependence. D will then meet each stratum in a $[\nu-1]$ with the possible exception of h separate strata, each of which D will meet in an $[\epsilon+\nu-1]$, where ϵ is the number of elementary divisors associated with the particular stratum. And if these meeting spaces of D and $\mathscr{R}$ do not completely define D, then D must also contain a singularity M of row dependence.*

Minimal indices of column dependence

17. The method of reflexion can now be applied to any stratum S, and it will give all information about the singular form N as well as the original form L. In fact let D meet S in the space S_{μ_0}, an $[\epsilon+\nu-1]$, and therefore admit a certain nest of spaces (S_μ) through successive reflexions

$$S_{\mu_0} \to A_{\mu_0} \twoheadrightarrow S_{\mu_1} \to A_{\mu_1} \twoheadrightarrow \dots. \tag{1}$$

Assume one or more Ψ_i to occur, else no new feature arises. Since D will now meet each stratum, we can take S to be the stratum C ($\alpha=0$) without loss of generality. The total meeting C_{μ_0} of D with C is then comprised in the blocks Θ and Ψ of the types

$$\Theta=\begin{bmatrix}\theta_1 & \theta_2 & \dots & \theta_{e-1} & . \\ \theta_0 & \theta_1 & \dots & \theta_{e-2} & \theta_{e-1}\end{bmatrix},\quad \Psi=\begin{bmatrix}\psi_0 & \dots & \psi_{n-1} \\ \psi_1 & \dots & \psi_n\end{bmatrix}. \tag{2}$$

On putting all the parameters of the upper rows equal to zero we have θ_0 and ψ_n, the first θ and last ψ of each such block, alone non-zero. All such θ_0, ψ_n therefore give the meeting C_{μ_0} of D with C. Proceeding as before we obtain A_{μ_0} from the same columns as C_{μ_0}, and then C_{μ_1} from the corresponding parameters, θ_0, θ_1 and ψ_n, ψ_{n-1}; and so on. The result will be a tableau which specifies the exact number of blocks Θ and Ψ, and the indices e or n, and the length of a chain initiated by any given point of C_{μ_0}; but it will fail to distinguish between the two types Θ and Ψ, or the indices e and n.

But on taking A_{λ_0}, the meeting of D with A, we may derive another sequence

$$A_{\lambda_0}\rightarrow C_{\lambda_0}\twoheadrightarrow A_{\lambda_1}\rightarrow C_{\lambda_1}\twoheadrightarrow\dots \tag{3}$$

in just the same way, which will specify the corresponding blocks Θ' and Ψ belonging to A. From the nests (A_μ) and (A_λ), so derived by the two processes (1) and (3), we can then select the common nest (A_ν), that is, the set of spaces common to corresponding members of the two nests: and the common nest will provide the tableau belonging to the Ψ alone.

For example, consider the eight-columned form

$$D=\begin{bmatrix}\theta_1 & . & \theta_0' & \theta_1' & \theta_2' & \psi_0 & \psi_1 & \psi_0' \\ \theta_0 & \theta_1 & \theta_1' & \theta_2' & . & \psi_1 & \psi_2 & \psi_1'\end{bmatrix}=[\Theta,\Theta',\Psi,\Psi']. \tag{4}$$

column number ... 1 2 3 4 5 6 7 8.

Here D meets C at C_{178} only, as given by θ_0, ψ_2, ψ_1': and meets A at A_{368} as given by θ_0', ψ_0, ψ_0'. The reflexive operations then give

$$\left.\begin{aligned}&C_{178}\rightarrow A_{178}\twoheadrightarrow C_{12768}\rightarrow A_{12768}\twoheadrightarrow C_{12768}\rightarrow \text{etc.,}\\ &A_{368}\rightarrow C_{368}\twoheadrightarrow A_{34678}\rightarrow C_{34678}\twoheadrightarrow A_{345678}\rightarrow \text{etc.,}\end{aligned}\right\} \tag{5}$$

both having become stabilized. The first set gives a nest of spaces A_{178}, A_{12768} with a tableau $\begin{smallmatrix}\times & \times \\ \times & \times \\ \times & \end{smallmatrix}$, obtained as before by counting the suffixes by columns. The second gives the nest A_{368}, A_{34678}, A_{345678} with the tableau $\begin{smallmatrix}\times & \times & \times \\ \times & \times & \\ \times & & \end{smallmatrix}$. The lengths of rows give the exponents 2, 2, 1 of e and n for C, and 3, 2, 1 of e' (say) and n for A. But the overlap in the nests yields first a single point a_8 from A_{178}, A_{368} and then a plane A_{678} from the next pair, and thereafter nothing new. This common nest segregates the singular Ψ, with a tableau $\begin{smallmatrix}\times & \times \\ \times & \end{smallmatrix}$ or $\begin{smallmatrix}6 & 7 \\ 8 & \end{smallmatrix}$, from the remaining row $\times\ \times$ of C and $\times\ \times\ \times$ of A, showing that C has one exponent $e=2$, A has one $e'=3$, while two indices n (1 and 2) are singular.

These indices n_i which specify the precise type of meeting between D and the whole set of strata are called after Kronecker the *minimal indices of column dependence*.

In the case when the stratum A has no non-singular meeting with D the process (3) yields the minimal indices at once as the row-partition of the tableau for (A_λ). Also, to find each set of non-singular exponents e, the process (1) must be applied to each of the h strata S in turn.

It should be remarked that any further blocks Θ belonging to strata distinct from A or C, as well as any of the Φ, have no effect upon the nests obtained in (1) and (3). Also the process applies as before when there are any number of blocks Θ or Ψ, belonging to A or C or to the singularity N. Thus we have proved the following result, generalizing on theorem 6.

THEOREM 10. *In the singular case the total meeting S_{μ_0} of D with any stratum S gives rise, by successive reflexion in B and D alternately, to a nest of spaces (A_ν) in any other stratum A. The conjugate partition of the consequent set of first differences yields the set of minimal indices n_i of column dependence (the same for each stratum), together with such of the exponents e_i as belong to S, in the case of h distinct strata.*

Answering to the complementary reflexive processes (5) we may use complementary tableaux—a left and a right:

$$\begin{array}{ccccc} & & & \times & \times \\ \times & \times & \times & & \\ \times & \times & & & \\ \times & & & & \end{array} \qquad \begin{array}{ccccc} & & & 2 & 1 \\ 3 & 4 & 5 & & \\ 6 & 7 & & & \\ 8 & & & & \end{array}, \qquad \begin{array}{ccccc} & & & \times & \times \\ \times & \times & \times & & \\ & & & \times & \times \\ & & & & \times \end{array} \qquad \begin{array}{ccccc} & & & 2 & 1 \\ 3 & 4 & 5 & & \\ & & & 6 & 7 \\ & & & & 8 \end{array}, \tag{6}$$

the former to be read by columns from the *left*, and the latter by columns from the *right*. The marks on the rows may be moved horizontally like beads on an abacus. Marks belonging to A and C are arranged in separate rows and columns, A left and C right; while those belonging to N are pushed home to the left under A or else to the right under C. In general the schemes are

$$\mathscr{T} = \begin{array}{cc} & T'_C \\ T_A & \\ T_N & \end{array} \quad \text{and} \quad \begin{array}{cc} & T'_C \\ T_A & \\ & T'_N \end{array} = \mathscr{T}', \tag{7}$$

where T_A is the tableau for the one or more blocks Θ belonging to A, while the accented T'_C, T'_N are right-tableaux for C and N, and T_N is the corresponding left-tableau for N. Had Θ' in (4) belonged to C, with θ'_0 leading off on the lower row, T'_C would have been $\begin{array}{ccc} & 2 & 1 \\ 5 & 4 & 3 \end{array}$. The columns, read respectively from the left and from the right, then indicate the nests of spaces (A_λ) and (C_μ) derived by the above reflexive processes; but, for this purpose, T'_C must be regarded as out of action in the left, and T_A in the right, tableau.

If x and y are two points chosen from the first column of T_N they must manifestly occupy different columns in T'_N, unless they belong to rows of equal length. Those elements x at the left end of the shortest rows of T_N will therefore appear in a later column of T'_N (read from left to right) than the y of a longer row. (Cf. $x = 8$, $y = 6$ in the illustration.) But the first column of $\mathscr{T}$ represents A_{λ_0}, while the first, the first two, the first three, etc. columns of $\mathscr{T}'$, read from right to left, represent the successive members of the nest (C_μ). Hence the first points of the spaces of (C_μ), taken in this ascending order, to lie in A_{λ_0} are the x: that is, the space of all these points x is uniquely determined by the intersection of A_{λ_0} and (C_μ).

Thus if there are exactly q least minimal indices n_i, there will be q points x, and a unique $[q-1]$ within the stratum A, derivable from the processes such as (5). This singles out for special prominence that ∞^q-fold *directrix* $\mathcal{N}_q$, belonging to the locus $\mathcal{N}$, which is of lowest order n_i and which meets each stratum in a $[q-1]$. The importance of this directrix was first noticed by Segre (1884*a*: cf. Edge 1931).

For example, in (4), $n_1 = 2$, $n_2 = 1$, so that ν the number of the n_i is 2; and $q = 1$ corresponding to the single Ψ' of least order. The locus $\mathcal{N}$ of D is given by the five parameters ψ_i, ψ'_i, and meets each stratum in a *line* ($\nu = 2$). It meets A at the line A_{68} (where $\psi_1 = \psi_2 = \psi'_1 = 0$) and C at C_{78}, and is a *ruled surface*. The ∞^1 directrix curves meeting A_{68} are *conics* except for a unique *straight line* ψ', the generating line $a_8 c_8$ of $\mathcal{R}$ in fact.

Had Ψ' been $\begin{bmatrix} \psi'_0 & \psi'_1 \\ \psi'_1 & \psi'_2 \end{bmatrix}$ with an extra 9th point its tableaux would have been $T_N = \begin{matrix} 6 & 7 \\ 8 & 9 \end{matrix} = T'_N$ and all the directrix curves would be conics with no distinctive points on the line A_{68} which they all meet.

Incidentally the above geometrical process throws light on the somewhat obscure algebraical fact that in reducing a matrix pencil to canonical form it is the minimal index of lowest value which first presents itself.

When D has a single minimal index n_1 and therefore a single curve $\mathcal{N}$, of order n_1, the canonical form of the corresponding space Ψ is explicable as follows: This curve meets two strata A and C in single points a_1 and c'_1 say. The osculating systems of these points interlock and at once give rise to chains $a_1 a_2 \ldots$, $c'_1 c'_2 \ldots$ of index n_1, where the set c'_i is the set $c_1 c_2 \ldots$ in reversed order. This process fixes the frame of reference which leads to the canonical form of Ψ. It may be verified by the methods of § 15.

For example, if $\mathcal{N}$ is a twisted cubic and $n_1 = 3$, then Ψ is the containing [3]: a_1 is the point where $\mathcal{N}$ cuts A, a_2 is where the tangent at a_1 to $\mathcal{N}$ cuts the osculating plane of c'_1, and a_3 is where the osculating plane at a_1 cuts the tangent at c'_1: and vice versa. Also $c'_1 \equiv c_3$, $c'_2 \equiv c_2$, $c'_3 \equiv c_1$. Similarly for higher values of n_1.

Similar remarks apply to the general case of several indices n_i. The interlocking of osculating systems of A and C determine the intermediate points of the canonical frame.

Minimal indices of row dependence

18. By transposing the matrix D to Δ and working with reflexions of primes instead of points we can derive the minimal indices m_i of row dependence for the singular pencil. But it is interesting to derive them directly in terms of points as before; and this is done by starting with the *total* reflexion of D in either A or C.

In fact, let A_d and C_d be these total reflexions of D; and let C_a be the reflexion of A_d through B to C, so that $A_d \to C_a$. Further, let

$$C'_d = (C_a, C_d) \tag{1}$$

denote the space common to C_a and C_d when they overlap or coincide. When they have no point common let $C'_d = 0$. It is convenient to denote this geometrical construction of C'_d from C_a and C_d by the implication sign $\supset$. Thus we shall have the sequence, $C_d \twoheadrightarrow A_d \rightarrow C_a \supset C'_d$, which may evidently be iterated:

$$C_d \twoheadrightarrow A_d \rightarrow C_a \supset C'_d \twoheadrightarrow A'_d \rightarrow C'_a \supset C''_d \twoheadrightarrow \dots. \tag{2}$$

Let this process be continued until either all successive spaces $C_d^{(r)}$ become identical or else are exhausted. In this way we obtain a nest of spaces

$$(C_d) \equiv C_d, C'_d, C''_d, \dots \tag{3}$$

of diminishing dimensions, which lead to the following theorem:

THEOREM 11. *The first differences of the dimensions, increased by unity, of the successive spaces in the nest* (C_d) *form a partition of an integer, whose conjugate partition gives the minimal indices* m_i *of column dependence, together with those exponents* e_i, *if any, which belong to the stratum* C.

Proof. As before, the reflexive process applies independently to each block Θ, Φ or Ψ of D. It will be shown that this has no effect upon any, except the Φ and those Θ which belong to C.

For any such $\Psi = \begin{bmatrix} \psi_0 & \dots & \psi_{n-1} \\ \psi_1 & \dots & \psi_n \end{bmatrix}$ we shall have $A_d = A_{12\dots n}$, as given by the n parameters of the upper row, and $C_d = C_{12\dots n}$ from the lower row; so that the process merely gives

$$C_d \twoheadrightarrow A_d \rightarrow C_d \supset C_d \twoheadrightarrow \dots.$$

Similarly for the Θ of § 13 (1) with $\alpha \neq 0$. Also for the case

$$\Theta' = \begin{bmatrix} \theta_0 & \theta_1 & \dots & \theta_{p-1} & \theta_p \\ \theta_1 & \theta_2 & \dots & \theta_p & . \end{bmatrix}, \tag{4}$$

which belongs to A we have, at once, $A_d = A_{12\dots e}$, $C_d = C_{12\dots p}$, where $e = p+1$, and so

$$C_d = C_{p!} \twoheadrightarrow A_{e!} \rightarrow C_{e!} \supset C_{p!} \twoheadrightarrow \dots, \tag{5}$$

with C_d unchanged throughout. But for the case C $(\alpha = 0)$ we have

$$\Theta = \begin{bmatrix} \theta_1 & \dots & . \\ \theta_0 & \dots & \theta_p \end{bmatrix},$$

and
$$C_d = C_{e!} \twoheadrightarrow A_{p!} \rightarrow C_{p!} \supset C_{p!} \twoheadrightarrow A_{(p-1)!} \rightarrow C_{(p-1)!} \supset C_{(p-1)!} \twoheadrightarrow \dots, \tag{6}$$

where a nest (C_d) is formed, consisting of a $[p]$ and lower spaces, diminishing by unit steps to the single point θ_0 and then vanishing.

Lastly for $\Phi = \begin{bmatrix} \phi_1 & \phi_2 & \dots & \phi_m & . \\ . & \phi_1 & \phi_2 & \dots & \phi_m \end{bmatrix}$ we evidently have $A_d = A_{12\dots m}$, $C_d = C_{23\dots, m+1}$, where as always the suffixes refer to the columns of the block. But $(C_{12\dots r}, C_{23\dots, r+1}) = C_{23\dots r}$. Hence

$$C_d = C_{23\dots, m+1} \twoheadrightarrow A_{12\dots m} \rightarrow C_{m!} \supset C_{23\dots m} \twoheadrightarrow \dots, \tag{7}$$

where again a nest (C_d) is obtained which falls one step at a time, until it is exhausted in m steps.

On combining all these results a nest (C_d) is formed, whose stepwise fall in dimensions will be entirely due to the Θ of C and the singular Φ; and the result follows.

COROLLARY. *By choosing* C *to be one of the strata which does not belong to the non-singular core of* D, *the process gives the set of minimal indices explicitly.*

For in this case α cannot be zero.

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Choice of canonical form

19. At first sight it would seem that this method of total reflexion supersedes the earlier method in giving all information about the exponents e_i more readily even in the non-singular case: but this is not so. The nest (C_d) terminates with a space C_μ which includes those parts of D given by the Ψ and Θ, excepting those Θ which belong to C. Similarly for any stratum S. Hence the process does not in general locate precisely the meeting of D and S already denoted by S_{μ_0}. Both methods of reflexion are in fact required to complete the geometrical theory.

For example, let $D = \begin{bmatrix} \phi_1 & \phi_2 & . & \phi' & . & \theta_1 & . & \\ . & \phi_1 & \phi_2 & . & \phi' & \theta_0 & \theta_1 & D_\mu \end{bmatrix}$, with $\mu+7$ columns in all, where D_μ denotes μ columns referring to neither Φ, A nor C. Here $m_1 = 2$, $m_2 = 1$, $e_1 = 2$, $C_d = C_{23567\mu}$, to use an obvious notation, the last suffix μ denoting a set of μ single suffixes. We should then have a nest

$$(C_d) \equiv C_{23567\mu}, \quad C_{26\mu}, \quad C_\mu,$$

where the first differences in numbers of suffixes are 3, 2, with a tableau $\begin{array}{cc} \times & \times \\ \times & \times \\ \times & \end{array}$, so that the indices m and e are 2, 2, 1. The corresponding nest for A (found by interchanging the roles of A and C) is

$$(A_d) \equiv A_{1246\mu}, \quad A_{26\mu}, \quad A_{6\mu},$$

where suffixes drop in steps 2, 1, so that the tableau is $\begin{array}{cc} \times & \times \\ \times & \end{array}$, and by the Corollary belongs to the m_i alone. This shows that the extra index in the C tableau indicates the single $e = 2$.

By the theorem, any further reflexions of C_μ and $A_{6\mu}$ merely repeat these spaces, which thus form a kind of permanent core to the nests. The extra dimension of $A_{6\mu}$ is due to the special contact of D on C. Had both strata A and C been chosen free of such meetings then their cores would be equal, and related by $A_\mu \rightarrow C_\mu$.

On taking the basis A, C free of such meetings we therefore obtain the permanent core in either, and consequently the permanent core D_μ of D which is the part (or whole) of D determined by the complete intersection of D and $\mathscr{R}$. Usually this does not account for the whole of D since the part due to the singular Φ is omitted. This can be found geometrically as follows. Let E denote any subspace of D supplementary to D_μ, so that E, D_μ together define D. Then the total reflexion of E must produce nests (C_e) and (A_e) which retain the minimal indices m_i only. The form Φ then belongs entirely to the subspace E which breaks up into further separate subspaces E_i of dimensions $(m_i - 1)$, according to the minimal indices. The canonical form assumed by Φ is due to selecting appropriate frames of reference within each such E_i. If m_1 is a single greatest index, the method obtains a unique canonical frame of m_1 points $d_1 d_2 \ldots d_{m_1}$, where d_1 is that single point of D derived by the overlap of the nests which initiates a chain of length m_1. When q of the m_i are greatest, q of the points d_i are independent and arbitrary within a fixed $[q-1]$ which initiates a chain of length m_1.

For example, if $E = \begin{bmatrix} \phi_1 & \phi_2 & . & \phi' & . \\ . & \phi_1 & \phi_2 & . & \phi' \end{bmatrix}$ which is a [3] say D_{124}, then this reflects to A_{124}, C_{235} respectively and gives rise to the nests C_{235}, C_2 and A_{124}, A_2, and a tableau $\begin{smallmatrix}\times & \times \\ \times & \end{smallmatrix}$. Each nest ends in a point, a_2 in the case of A: and this defines d_2 of D. By reflexions all the points a_1, a_2, a_3, c_1, c_2, c_3 are successively obtained from a_2. These define d_1, d_2 uniquely by $a_1 \twoheadrightarrow d_1 \twoheadrightarrow c_2$, $a_2 \twoheadrightarrow d_2 \twoheadrightarrow c_3$; but d_4, depending on the lower index m_2 $(=1)$ is one of ∞^1 possibilities.

The vectors of apolarity

20. In the Kronecker theory of singular pencils of matrices certain non-zero row and column vectors u and x arise which annihilate the pencil identically: that is

$$u(\rho D_1 + \sigma D_2) = 0, \quad (\rho D_1 + \sigma D_2)\, x = 0, \tag{1}$$

for all values of ρ and σ. To interpret these conditions geometrically we write them as

$$[\rho u, \sigma u]\begin{bmatrix} D_1 \\ D_2 \end{bmatrix} = 0, \quad [D_1, D_2]\begin{bmatrix} \rho x \\ \sigma x \end{bmatrix} = 0, \tag{2}$$

which imply that a certain prime $U = [\rho u, \sigma u]$ belonging to the scroll $\mathscr{R}$ contains the space D, and a certain point $X = \{\rho x, \sigma x\}$, also belonging to $\mathscr{R}$, lies in a space $D'' = [D_1, D_2]$. This latter must not be confused with the space $D' = [D_1', D_2']$ obtained by transposing the $k \times k$ matrices D_1 and D_2.

When D is in canonical form the above u breaks up into a set of μ independent vectors, one for each minimal index m_i. Such a component vector takes the form

$$\omega = [\sigma^p, -\rho\sigma^{p-1}, \rho^2\sigma^{p-2}, \ldots, (-)^p \rho^p] \quad (p = m_i), \tag{3}$$

which annihilates the corresponding part M_i of the canonical form of $\rho D_1 + \sigma D_2$. Similarly for ν component vectors of the vector x, with $p = n_i$. These vectors u and x are the minimal vectors, and their components, ω or its dual, have been called the vectors of apolarity.

Now corresponding to m_i is a certain space E_i of D external to $\mathscr{R}$. This reflects to A_e and C_e, say, of A and C, which together comprise a portion $\mathscr{R}_e$ of $\mathscr{R}$: that is, every generating line g of $\mathscr{R}_e$ meets at least one of A_e and C_e. Each of E_i, A_e, C_e, $\mathscr{R}_e$ lies in the $[2m_i + 1]$ which is otherwise expressed by the portion Φ_i of D already considered.

From (2) it follows that within the space Φ_i there is a prime $\pi = [\rho\omega, \sigma\omega]$ which contains E_i for all values of ρ, σ. That is, the vectors of apolarity, belonging to m_i, are given by the tangent primes of a certain *cone* for which E_i is the vertical region. Also (3) shows that each of these primes, which belong to, and therefore touch $\mathscr{R}_e$, osculates, to a degree m_i, a certain base curve N' at the point of contact s on $\mathscr{R}_e$.

This curve of $\mathscr{R}_e$ is given by the relations

$$N' \equiv \operatorname{loc}\{\sigma\xi, -\rho\xi\} = \operatorname{loc} s, \tag{4}$$

where

$$\xi = \left\{\rho^p, p\rho^{p-1}\sigma, \binom{p}{2}\rho^{p-2}\sigma^2, \ldots, \sigma^p\right\} \quad (p = m_i). \tag{5}$$

This point $s = \{\sigma\xi, -\rho\xi\}$, where the $p+1$ components of ξ are the terms in the expansion of $(\rho+\sigma)^p$, lies on the prime π, as is at once apparent from the vanishing series

$$\pi s = [\rho\omega, \sigma\omega]\{\sigma\xi, -\rho\xi\} = 0.$$

Also if $s^{(q)}$ denotes $\partial^q s/\partial \rho^q$ we find, by Leibniz' theorem, that

$$\pi s^{(q)} = [\rho\omega, \sigma\omega]\{\sigma\xi^{(q)}, -\rho\xi^{(q)} - q\xi^{(q-1)}\} = -q\sigma\omega\xi^{(q-1)},$$

which vanishes with the series $\omega\xi^{(q-1)}$ $(q = 1, 2, \ldots, p)$ by a well-known property of the binomial coefficients.* Hence π contains the $p+1$ points s, s', ..., $s^{(p)}$ and has contact of order $p = m_i$ with the curve N' at s.

Conversely the point s describes the curve N' which lies in a $[p+1]$. By what has just been proved the osculating $[p]$ of N' at s is the intersection of π with this containing space $[p+1]$. The space sE_i obtained by joining s to the whole of E_i lies in π and is the generator of the cone: that is, the cone is described by letting s describe the curve N'.

Corresponding to the μ indices m_i there are μ such cones, and by compounding them together in a more general cone with E for vertical region, we derive an interpretation of the vector u in (1) from a tangent prime to a certain ∞^μ region $\mathcal{N}'$ of the type considered in § 15.

For the case of a single minimal index n_i of column dependence an interpretation, dual to the above, can be given. The vector $[\rho x, \sigma x]$ is then a point on a certain normal curve N of order n_i+1 which is the intersection of $\mathscr{R}$ with the primary space $[D_1, D_2]$. The vector x itself can be regarded as the g-generator of $\mathscr{R}$ through such a point. For ν indices n_i the curve N becomes a rational ν-fold region.

Latent loci

21. It is readily verified that the matrix

$$H = \rho\begin{bmatrix} 1 & . & . & \cdots & . \\ a & b & . & \cdots & . \\ a^2 & 2ab & b^2 & \cdots & . \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a^p & pa^{p-1}b & . & \cdots & b^p \end{bmatrix} \quad (b \neq 0),$$

transforms a column vector $x = \{1, \theta, \theta^2, \ldots, \theta^p\}$ to a similar set $\xi = \{1, \phi, \phi^2, \ldots, \phi^p\}$, where $\phi = a + b\theta$ and $\rho\xi = Hx$. Conversely it is also readily verified that this is the only matrix which transforms x to ξ for all values of θ.

The points x and ξ then lie on the same curve N, which is therefore a *latent curve* of the transformation. Excluding the trivial case where $a = 0$, $b = 1$, two cases arise:

$$\text{(i)}\ \ a \neq 0,\ b = 1, \qquad \text{(ii)}\ \ b \neq 1.$$

In (i) H has one latent root, which is unity, and one elementary divisor whose index is $p+1$. Corresponding therefore to each elementary divisor with an index e exceeding unity of a general matrix there is a certain rational normal curve N of order $e-1$ which is latent for the transformation and lies in the corresponding subspace $[e-1]$.

In (ii) the matrix H has $p+1$ latent roots ρ, ρb, ..., ρb^p which are in geometrical progression, so that H can be reduced to purely diagonal form. In effect we can take $a = 0$ and $\phi = b\theta$. This gives the following result:

* This expression for point and osculating prime of a normal curve is due to Clifford (1878).

THEOREM 12. *When D meets $\mathscr{R}$ at a set of q ordinary points such that the cross ratios of the q corresponding transversal lines $a_\lambda b_\lambda c_\lambda d_\lambda$ are numbers in geometrical progression, then a latent N curve exists of order $q-1$. Such a curve also exists when D touches $\mathscr{R}$ with $(q-1)$ fold contact at a point. Otherwise no such latent N curve exists.*

One such N exists for each index e exceeding unity, and for each distinct set of latent roots in geometrical progression when $e=1$.

Latent surfaces and higher loci of this type $\mathscr{N}$, already considered, may be dealt with in the same way. They would also serve to illustrate geometrically the algebraic processes involved in passing from a semi-reduced to a completely reduced canonical form of matrix.

Transcendental latent curves exist in the general case: for example, the curve loc $\{A^\theta x\}$, where θ is the parameter and A is an $n\times n$ matrix is latent for the collineation $y=Ax$, points of parameter θ and $\theta+1$ being corresponding points. Such a curve always exists when A is non-singular, and there is a curve of this type through *every* pair of corresponding points, as shown by taking $\theta=0$, $\theta=1$. But the curve is algebraic and not transcendental in special cases, as when integral powers of the latent roots exist which are all equal.

An interesting discussion of case (i) above is given by Enriques (1918) who also appends a short historical account of the whole theory.

INVARIANTS OF THE MATRIX PENCIL

22. There is an invariant theory associated with the collineations $(ijkl)$ of § 8. I here state the main results without proof (Turnbull 1942). The 24 collineations belong to three pairs of matrices § 8 (1) and three characteristic equations each of order k. If α is a root of one such equation, then $1-\alpha$ and $\alpha/(\alpha-1)$ are respectively the roots of the other two.

If
$$X_0\theta^k - X_1\theta^{k-1}+\ldots+(-)^k X_k = 0,$$

is one of these equations, its coefficients X_i are rational integral invariants of the four medials A, B, C, D. In fact $X_i=\Sigma(-)^{ij}\Delta_i$, where Δ_i denotes the $4k$-rowed determinant

$$\begin{vmatrix} A_i & B & C & . & . \\ . & . & C & D & A_j \end{vmatrix}$$

and the sum extends to all the determinantal permutations of $A=A_iA_j$ $(i+j=k)$, that is to $\binom{k}{i}$ terms. Each of A, B, C, D has k columns. Exactly $k-2$ of the X_i are irreducible, and $X_0=(BC)(DA)$, $X_k=(AB)(CD)$.

On permuting A, B, C, D in all possible ways in Δ_i two further sets Y_i and Z_i alone are found, accounting for the two further characteristic equations. On writing

$$X_i=(ABCD)_i,$$

then
$$Y_i=(ADBC)_i \quad\text{and}\quad Z_i=(ACDB)_i.$$

Moreover, the sets X, Y, Z are connected by the linear relations

$$X = QY, \quad Y = QZ, \quad Z = QX, \quad Q^3 = I,$$

where

$$Q = \begin{bmatrix} \cdots & \cdots & \cdots & \cdots \\ . & . & 1 & \vdots \\ . & -1 & 2 & \vdots \\ 1 & -1 & 1 & \vdots \end{bmatrix}$$

a triangular $k+1 \times k+1$ matrix of binomial coefficients whose cube is the unit matrix.

The roots of the characteristic equations and their reciprocals are the six sets of cross ratios on the k latent lines of the collineations. One root is -1 if one of ΣX, ΣY, ΣZ vanishes. Hence *the necessary and sufficient condition for four* $[k-1]$*'s in* $[2k-1]$ *to cut one of their transversal lines harmonically is*

$$\Sigma_i X_i \,.\, \Sigma_i Y_i \,.\, \Sigma_i Z_i = 0.$$

If $k = 2$, the condition for one transversal of four skew lines in [3] to be cut harmonically is

$$(-p+2q+2r)\,(2p-q+2r)\,(2p+2q-r) = 0,$$

where $p = (BC)\,(AD)$, $q = (CA)\,(BD)$, $r = (AB)\,(CD)$ in terms of the mutual moments (BC), etc. of the lines A, B, C, D.

The condition $\surd p + \surd q + \surd r = 0$ implies that the cross ratios on the two transversals are equal, which holds when either the line D touches the quadric $\mathscr{R}$ through A, B and C or else is a generator of $\mathscr{R}$.

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